



Contents lists available at ScienceDirect

# Journal of Mathematical Economics

journal homepage: [www.elsevier.com/locate/jmateco](http://www.elsevier.com/locate/jmateco)



## Risk aversion and the elasticity of substitution in general dynamic portfolio theory: Consistent planning by forward looking, expected utility maximizing investors

Richard Kihlstrom <sup>a,b,\*</sup>

<sup>a</sup> Finance Department, The Wharton School, University of Pennsylvania, Philadelphia, PA 19104-6367, United States

<sup>b</sup> The Economics Department, University of Pennsylvania, United States

### ARTICLE INFO

#### Article history:

Received 27 April 2007  
Received in revised form 6 August 2008  
Accepted 19 August 2008  
Available online xxx

#### Keywords:

Dynamic portfolio theory  
Financial market equilibrium  
Asset pricing

### ABSTRACT

Using the measure of risk aversion suggested by Kihlstrom and Mirman [Kihlstrom, R., Mirman, L., 1974. Risk aversion with many commodities. *Journal of Economic Theory* 8, 361–388; Kihlstrom, R., Mirman, L., 1981. Constant, increasing and decreasing risk aversion with many commodities. *Review of Economic Studies* 48, 271–280], we propose a dynamic consumption–savings–portfolio choice model in which the consumer–investor maximizes the expected value of a non-additively separable utility function of current and future consumption. Preferences for consumption streams are CES and the elasticity of substitution can be chosen independently of the risk aversion measure. The additively separable case is a special case. Because choices are not dynamically consistent, we follow the “consistent planning” approach of Strotz [Strotz, R., 1956. Myopia and inconsistency in dynamic utility maximization. *Review of Economic Studies* 23, 165–180] and also interpret our analysis from the game theoretic perspective taken by Peleg and Yaari [Peleg, B., Yaari, M., 1973. On the existence of a consistent course of action when tastes are changing. *Review of Economic Studies* 40, 391–401]. The equilibrium of the Lucas asset pricing model with i.i.d. consumption growth is obtained and the equity premium is shown to depend on the elasticity of substitution as well as the risk aversion measure. The nature of the dependence is examined. Our results are contrasted with those of the non-expected utility recursive approach of Epstein–Zin and Weil.

© 2008 Elsevier B.V. All rights reserved.

### 1. Introduction

Until recently the model used to analyze a consumer–investor’s dynamic consumption–saving and portfolio choices was the “additively separable” model in which the consumer–investor is assumed to maximize

$$U(\{c_t\}_{t=1}^{\infty}) = E \left[ \sum_{t=1}^{\infty} \beta^t u(\tilde{c}_t) \right], \quad (1)$$

the expected value of the discounted sum of utilities of per-period random consumption,  $\tilde{c}_t$ . Early analyses of this model were provided in the papers by Levhari and Srinivasan (1969), Merton (1969, 1971), Samuelson (1969) and Hakansson (1970). Merton (1973) used this framework to obtain a dynamic asset pricing model. The implications of Merton’s asset pricing model

\* Tel.: +1 215 898 4378; fax: +1 215 898 6200.  
E-mail address: [kihlstro@wharton.upenn.edu](mailto:kihlstro@wharton.upenn.edu).

were clarified in Breeden (1979). In his pioneering (1978) paper, Lucas also used the additively separable model to construct a dynamic asset pricing model. The Lucas and Merton models provide the foundation for much of the subsequent work on dynamic asset pricing. In a significant part of this research, the utility function,  $u(\cdot)$ , of per-period consumption is assumed to be in the constant relative risk averse (CRRA) class. If  $u(\cdot)$  is CRRA with relative risk aversion  $\alpha$ , the utility function (1) of the certain consumption stream  $\{c_t\}_{t=1}^{\infty}$  is in the CES class with elasticity of substitution:

$$\sigma = \frac{1}{\alpha}.$$

This, of course, means that, when this model is used to analyze asset pricing, the impact of risk aversion, as measured by  $\alpha$ , cannot be separated from the impact of a change in the elasticity of substitution of the ordinal preferences for consumption streams  $\{c_t\}_{t=1}^{\infty}$ .

Recent contributions by Epstein (1988), Epstein and Zin (1989, 1990, 1991) and Weil (1989, 1990), have extended the analysis of the Lucas model to particular cases of recursive, non-additive preferences of the type introduced in Kreps and Porteus (1978, 1979a, b). The motivation for the Epstein–Zin and Weil (EZW) extensions was the inability of the Lucas model with CRRA preferences to explain the size of the equity premium without assuming values of  $\alpha$  that were widely regarded as “too large.” This “equity premium puzzle” was pointed out in Mehra and Prescott (1985) and Grossman et al. (1987) which built on the earlier contribution in Grossman and Shiller (1981).

By using recursive, non-additive Kreps–Porteus preferences EZW were able to choose the risk aversion parameter independently of the elasticity of substitution. The additional degree of freedom in the EZW formulation would appear to enhance the ability of the Lucas model to provide an explanation for a wide variety of consumption-saving, portfolio choice and asset pricing phenomena. As noted in the introduction to Epstein and Zin (1991), “the disentangling of risk aversion from the elasticity of substitution is a problem that has been highlighted by the empirical literature on the behavior of asset returns and consumption over time. Representative agent optimizing models have not performed well empirically (see, among others, Hansen and Singleton, 1983; Mehra and Prescott, 1985; Grossman et al., 1987). One possible reason for this poor performance is that the maintained specification of preferences is too rigid.” But as Weil (1989) points out “the solution to the equity premium puzzle documented by Mehra and Prescott (1985) cannot be found by simply separating risk aversion (from) intertemporal substitution. If the dividend growth process is i.i.d., the risk-premium, when appropriately defined, is independent of the intertemporal elasticity of substitution, and thus is the same whether or not the time-additive, expected utility restriction is imposed. When the dividend growth process is non-i.i.d., relaxing the parametric restriction adds, for plausible parameter values, a risk free rate puzzle to Mehra and Prescott’s equity premium puzzle.” Kocherlakota (1990) demonstrates an even stronger result. In analyzing the Lucas asset pricing model he assumes Epstein–Zin preferences and that “the growth rate of the aggregate endowment is i.i.d.” Using this model he demonstrates that “an econometrician with data on asset prices and aggregate consumption cannot separately identify” the elasticity of substitution and the relative risk aversion. He concludes that the Epstein–Zin preferences have “no more explanatory power than” the preferences represented by the utility function in (1). The inability of the EZW framework to add explanatory power beyond that of the additively separable model in the i.i.d. case is rooted in the observation made in a number of papers that, when the returns to the risky assets are i.i.d., the EZW optimal portfolio depends only on the risk aversion measure and is independent of the elasticity of substitution. See for example, Swensson (1989) and Bhamra and Uppal (2006).

In the EZW approach, consumer-investors do not maximize expected utility. Epstein and Zin (1989) briefly discussed but rejected an alternative approach, based on expected utility maximization, to the problem of using different parameters to measure risk aversion and the elasticity of substitution. The alternative approach that they argued was inappropriate for use in a dynamic setting is based on the discussion of risk aversion with many commodities in Kihlstrom and Mirman (1974, 1981). There is, indeed, a difficulty that arises when one attempts to implement the Kihlstrom–Mirman approach in a non-additively separable dynamic model of the type envisioned by Epstein and Zin. Specifically, dynamic consistency requires that the best current choice depends on choices made in the past. Thus, memory of past choices is a necessary condition for dynamic consistency. It was not simply the fact that the memory of the past was necessary that Epstein and Zin found troubling. Indeed, memory of the past is required even in additively separable dynamic models that assume preferences are influenced by habits; see, for example, Constantinides (1990), Abel (1990, 1999) and Campbell and Cochrane (1999). What did prove objectionable and decisively so was the fact that, because of the usual discounting assumptions, memory of the very distant past would become most important. In recent working papers, Eden (2005) and Van den Heuvel (2007) have investigated the approach rejected by Epstein and Zin. Under the discounting assumptions of Van den Heuvel’s paper, memory of the distant past is shown to become unimportant.

In the current paper, we also reconsider the possibility of implementing an alternative dynamic model based on the Kihlstrom–Mirman expected utility maximizing approach. In our alternative formulation, preferences change over time since the consumer-investor’s only concern is with current and future consumption. Because the consumer-investor ignores past consumption, we refer to him as having “forward looking” von Neumann–Morgenstern preferences. In the dynamic model considered below, the consumer-investor’s current consumption-saving and portfolio choices at each point in time maximize his expected utility of current and future consumption. This implies that the consumer-investor’s current choices are independent of past consumption. This independence of past consumption implies a dynamic inconsistency. The current choices are not the ones the consumer-investor would have committed himself to make if such a commitment had been possible at an earlier point in time.

The dynamic inconsistency we are forced to deal with is similar to but distinct from that which arises in models of hyperbolic discounting such as those first considered by [Strotz \(1956\)](#) and recently reconsidered in the work of [Laibson \(1997\)](#), [Harris and Laibson \(2001\)](#) and [Luttmer and Mariotti \(2003, 2007\)](#). In analyzing the model we propose, we follow the “consistent planning” approach of [Strotz \(1956\)](#) and assume that, when making his current choice, the consumer-investor will “take account of future disobedience.” This consistent planning approach was also used by [Pollak \(1968\)](#) and by [Phelps and Pollak \(1968\)](#). In the dynamic consumption-savings, portfolio model we consider, when the consumer-investor makes his current choices he recognizes that his future choices will not be the ones he would currently like to commit himself to make in the future. Thus, the consumer-investor chooses a consumption plan for the future that is, as [Strotz](#) asserted, “the best plan among those he will actually follow.” Our approach can also be interpreted from the perspective of [Peleg and Yaari \(1973\)](#) and is also similar to that taken in the literature on durable goods monopoly and the Coase conjecture; see, for example, [Coase \(1972\)](#), [Stokey \(1979\)](#) and [Bulow \(1982\)](#). In following [Peleg and Yaari](#) and the durable goods monopoly literature, we view the current consumer-investor as a leader in a leader follower game in which the followers are the same consumer-investor at future time periods. His current choices are a best response to the choices he knows he will want to make in the future. The result is a Nash equilibrium of a Stackelberg game in which each “player” is the consumer-investor at a particular consumption period.

We refer to our approach as one of “consistent planning” by a “forward looking” expected utility maximizing consumer-investor. Using this approach we follow [Mehra–Prescott](#) and [Epstein–Zin](#) and [Weil](#) and investigate the risk premium implied by equilibrium in the Lucas asset pricing model with CES preferences when consumption growth is i.i.d. We do find that, in contrast to the [Epstein–Zin](#) and [Weil](#) approach, the risk premium obtained from our approach is affected by the elasticity of substitution as well as the risk aversion measure. It thus does not suffer from the criticism of [Kocherlakota](#) and of [Weil](#) himself. In fact, one of the main results of this paper, which was pointed out by [Kocherlakota](#), is that our model yields a higher risk premium than the standard additively separable model when and only when the elasticity of substitution in our model is exceeded by that of the additively separable model. In spite of this, it must be added that our approach may not have great promise for explaining the [Mehra–Prescott](#) equity premium puzzle when consumption growth is assumed to be i.i.d. Indeed, [Mehra and Prescott](#) suggested that there is good reason to be pessimistic about the promise of models based on CES preferences to imply a large equity premium. In their paper, they asserted that “We doubt whether non-time-additivity separable preferences will resolve the puzzle, for that would require consumptions near in time to be poorer substitutes than consumptions at widely separated dates.”

[Labadie \(1986\)](#) has also investigated the implications of the [Kihlstrom–Mirman](#) approach to risk aversion for the equity premium obtained in a Lucas asset pricing model equilibrium. Her work which was motivated by [Diamond and Stiglitz \(1974\)](#) avoided the problem of dynamic inconsistency by using an overlapping generations model in which the consumer-investors lived for two periods.

### 1.1. Outline of the paper

We introduce the [Kihlstrom–Mirman](#) approach to risk aversion with many commodities by considering the consumption-saving, portfolio problem of a consumer-investor who lives for two periods. The [Kihlstrom–Mirman](#) approach assumes that the consumer-investor maximizes expected utility. Our discussion of the [Kihlstrom–Mirman](#) approach based on expected utility maximization is followed by a description of the [Epstein–Zin](#), [Weil](#), [Kreps–Porteus](#) approach to the two-period consumption-saving portfolio problem. We focus on the fact that their approach is not based on expected utility maximization and emphasize the contrast between the two approaches.

We then proceed to introduce our analysis of the dynamic model of forward looking von Neumann–Morgenstern preferences by considering the three-period case. In this section we focus on the dynamic inconsistency. But the three-period case also provides a useful setting for the introduction of the consistent planning approach.

Next we consider the infinite horizon problem and contrast the equilibrium that results from the forward looking von Neumann–Morgenstern preferences with that obtained from the [EZW](#) approach. Finally, we use the forward looking von Neumann–Morgenstern preferences to obtain an equilibrium of the Lucas asset pricing model. We contrast this with the equilibrium of the Lucas asset pricing model obtained using the [EZW](#) approach and with the equilibrium of the Lucas model obtained using additively separable preferences. One of our main results is [Proposition 4](#) which asserts that the equity premium resulting from the equilibrium obtained with the forward looking von Neumann–Morgenstern preferences exceeds that obtained from the additively separable model only when the elasticity of substitution falls below the level it is restricted to take on in the additively separable case. As mentioned in [Section 1](#), this result is due to [Kocherlakota](#). [Proposition 2](#) describes the conditions under which expected utility is finite in the infinite horizon model. [Proposition 5](#) demonstrates that savings increase (decrease) with risk aversion if the elasticity of substitution is less (greater) than one. This result extends a result obtained in [Kihlstrom and Mirman \(1974\)](#) and [Diamond and Stiglitz \(1974\)](#).

## 2. The two-period case

### 2.1. The Kihlstrom–Mirman approach to risk aversion

In this section, we illustrate the use of the [Kihlstrom–Mirman](#) approach to risk aversion with many commodities by considering a two-period consumption-saving portfolio problem in which the consumer-investor chooses the fraction,  $c$ , of

his initial wealth,  $W$ , to consume in the first consumption period. First period savings, which equal  $(1 - c)W$  can be invested in a safe and a risky asset. Thus, the consumer-investor must also choose the fraction,  $\gamma$ , of savings to invest in the risky asset. The safe asset return is  $r_f$ . The realized return on the risky asset is  $r_m$ . The random variable that denotes the risky asset return is  $\tilde{r}_m$ . It is useful to define the realized excess return on the risky asset as

$$x_m = r_m - r_f.$$

This is, of course, simply the realization of the random variable:

$$\tilde{x}_m = \tilde{r}_m - r_f.$$

Second period consumption is then

$$W[1 - c][r_f + \gamma x_m].$$

Following von Neumann–Morgenstern, we assume that the consumer-investor solves this two-period consumption-saving portfolio problem by simply maximizing expected utility and choosing

$$(\hat{c}, \hat{\gamma}) = \arg \max_{(c, \gamma)} E[U(cW, W[1 - c][r_f + \gamma \tilde{x}_m])]$$

where

$$U(C_1, C_2)$$

is some strictly concave function of the two-period consumption stream:

$$(C_1, C_2).$$

This is the standard general approach to the two-period problem. This was also the approach taken in Kihlstrom and Mirman (1974) and in a related paper, Diamond and Stiglitz (1974), in the same journal. In those papers, there was, however, no riskless asset and hence no portfolio choice problem.

In what follows we describe the Kihlstrom–Mirman approach to risk aversion with many commodities and indicate how it can be applied to study the effect of risk aversion on the  $(\hat{c}, \hat{\gamma})$  choice in the two-period problem. The general Kihlstrom–Mirman approach to risk aversion with many goods restricts the risk aversion comparisons of utility functions to those which represent the same ordinal preferences. Thus, the risk aversion of two utility functions  $U^1$  and  $U^2$  of two-period consumption streams are compared only if they are related by a monotonically increasing transformation. The utility function  $U^1$  is defined to be more risk averse than  $U^2$  if

$$U^1(C_1, C_2) = h(U^2(C_1, C_2))$$

and the transformation  $h$  is strictly concave.

To define the absolute or relative risk aversion of a utility function of consumption streams:

$$(C_1, C_2)$$

attention is restricted to those ordinal preferences for which there exists a “least concave representation,”  $U^0$ , for which every concave representation of the preferences is a concave transformation of  $U^0$ . This definition of “least concave” was introduced in Debreu (1976). When the preferences are homothetic, the least concave representation is homogeneous of degree one. Thus, when the approach is applied to CES ordinal preferences,

$$U^0(C_1, C_2) = (C_1^\rho + \beta C_2^\rho)^{1/\rho},$$

if  $\rho \neq 0$ . In the Cobb-Douglas case corresponding to  $\rho = 0$ ,

$$U^0(C_1, C_2) = C_1^{(1/(1+\beta))} C_2^{(\beta/(1+\beta))}.$$

If  $U(C_1, C_2)$  is a strictly concave representation of some ordinal preferences for which a least concave least concave representation,  $U^0$ , exists then

$$U(C_1, C_2) = h(U^0(C_1, C_2))$$

where  $h(\cdot)$  is monotone increasing and strictly concave. The absolute risk aversion of  $U$  at  $(C_1, C_2)$  is defined to be

$$-\frac{h''(U^0(C_1, C_2))}{h'(U^0(C_1, C_2))}.$$

The relative risk aversion of  $U$  at  $(C_1, C_2)$  is defined to be

$$-\frac{h''(U^0(C_1, C_2))U^0(C_1, C_2)}{h'(U^0(C_1, C_2))}.$$

This means that for CES ordinal preferences with  $\rho \neq 0$  the relative risk aversion of the representation:

$$U^\alpha(C_1, C_2) = \frac{(C_1^\rho + \beta C_2^\rho)^{(1-\alpha)/\rho}}{1-\alpha}$$

is  $\alpha$  which we have assumed to be positive and unequal to one. The case of  $\alpha = 1$  is of course

$$U^\alpha(C_1, C_2) = \log((C_1^\rho + \beta C_2^\rho)^{(1/\rho)}).$$

In the Cobb-Douglas case corresponding to  $\rho = 0$ , the relative risk aversion of

$$U^\alpha(C_1, C_2) = \frac{C_1^{(1-\alpha)/(1+\beta)} C_2^{\beta(1-\alpha)/(1+\beta)}}{1-\alpha}$$

is  $\alpha$ , where once again we have assumed that  $\alpha \neq 1$ . In the Cobb-Douglas case when  $\alpha = 1$ ,

$$U^\alpha(C_1, C_2) = \log C_1 + \beta \log C_2.$$

## 2.2. Applying the Kihlstrom–Mirman approach to the consumer-investor’s problem

In the Kihlstrom–Mirman approach,  $U^0$  is the risk neutral representation of the preferences even when  $U^0$  is not homogeneous of degree one. In the present section and in the remainder of the discussion in this section we will, however, assume that  $U^0$  is homogeneous of degree one so that the preferences are homothetic. In this case, when the risk neutral consumer-investor solves the two-period consumption-saving portfolio problem and chooses

$$(\hat{c}, \hat{\gamma}) = \arg \max_{(c, \gamma)} E[U^0(cW, (W[1-c][r_f + \gamma \tilde{x}_m]))],$$

the value function  $J_0(\cdot)$  defined by

$$J_0(W) = \max_{(c, \gamma)} E[U^0(cW, W[1-c][r_f + \gamma \tilde{x}_m])]$$

is

$$J_0(W) = \kappa_0 W$$

where

$$\kappa_0 = \max_{(c, \gamma)} E[U^0(c, [1-c][r_f + \gamma \tilde{x}_m])].$$

The linearity of  $J_0$  in  $W$  implies risk-neutrality in the face of wealth risks. The risk-neutrality of the value function is inherited from the risk-neutrality of  $U^0$ .

Also when  $U^0$  is homogeneous of degree one, if the consumer-investor chooses

$$(\hat{c}, \hat{\gamma}) = \arg \max_{(c, \gamma)} E \left[ \left( \frac{1}{1-\alpha} \right) (U^0(cW, W[1-c][r_f + \gamma \tilde{x}_m]))^{1-\alpha} \right] = \arg \max_{(c, \gamma)} (E[(U^0(c, [1-c][r_f + \gamma \tilde{x}_m]))^{1-\alpha}])^{1/(1-\alpha)} \quad (2)$$

then his value function  $J_\alpha(\cdot)$  defined by

$$J_\alpha(W) = \max_{(c, \gamma)} E \left[ \left( \frac{1}{1-\alpha} \right) (U^0(cW, W[1-c][r_f + \gamma \tilde{x}_m]))^{1-\alpha} \right]$$

is

$$J_\alpha(W) = \kappa_\alpha W^{1-\alpha}$$

where

$$\kappa_\alpha = \max_{(c, \gamma)} \frac{E[(U^0(c, [1-c][r_f + \gamma \tilde{x}_m]))^{1-\alpha}]}{1-\alpha}.$$

and  $\alpha$  is the relative risk aversion of  $J_\alpha$ . In this case, the value function  $J_\alpha(\cdot)$  again inherits the risk aversion of

$$U^\alpha(C_1, C_2) = \left( \frac{1}{1-\alpha} \right) (U^0(C_1, C_2))^{1-\alpha}.$$

Following Epstein and Zin we can define the “certainty equivalent utility” as

$$V(W) = \max_{(c, \gamma)} (E[(U^0(cW, W[1-c][r_f + \gamma \tilde{x}_m]))^{1-\alpha}])^{1/(1-\alpha)}.$$

Clearly the certainty equivalent utility is linear in wealth since the linear homogeneity of  $U^0(\cdot, \cdot)$  implies that

$$V(W) = vW$$

where

$$v = (E[(U^0(\hat{c}, [1 - \hat{c}][r_f + \hat{\gamma}\tilde{x}_m]))^{1-\alpha}])^{1/(1-\alpha)} = ((1 - \alpha)\kappa_\alpha)^{1/(1-\alpha)}$$

and where  $(\hat{c}, \hat{\gamma})$  is given by (2).

In spite of the argument just outlined, it is true that, in the two-period consumption-savings portfolio choice problem, the risks the consumer-investor faces are risks involving second period consumption and  $U^0(C_1, C_2)$  is not linear in  $C_2$ . Thus, when choosing a portfolio, the consumer investor who maximizes the expected value of  $U^0(C_1, \tilde{C}_2)$  is not risk-neutral. Specifically in the CES case with

$$\rho \neq 0, \\ U^0(C_1, C_2) = (C_1^\rho + \beta C_2^\rho)^{1/\rho}$$

and with

$$\rho = 0, \\ U^0(C_1, C_2) = C_1^{1/(1+\beta)} C_2^{\beta/(1+\beta)}.$$

In both cases,

$$U_{C_2, C_2}^0(C_1, C_2) < 0$$

so that  $U^0(C_1, C_2)$  is strictly concave in  $C_2$ .

In general, it is difficult to get comparative static results for  $(\hat{c}, \hat{\gamma})$  as the risk aversion of the representation changes. It is true however that for  $c$  fixed the expected utility maximizing  $\gamma$  decreases if risk aversion or relative risk aversion increases. Also, if the elasticity of substitution is uniformly less (greater) than one, then for  $\gamma$  fixed the expected utility maximizing  $c$  decreases (increases) if risk aversion or relative risk aversion increases. The former result is a corollary to the well-known results proven by Arrow (1971) and Pratt (1964). The latter result is a consequence of a result proven in Kihlstrom and Mirman (1974).

### 2.2.1. The CES case

We only consider the case of  $\alpha \neq 1$ . If, in this case, the elasticity of substitution,  $\sigma$ , is unequal to 1, then

$$\rho \neq 0$$

and

$$V(W) = \max_{(c, \gamma)} (E[(cW)^\rho + \beta[W(1 - c)(r_f + \gamma\tilde{x}_m)]^\rho]^{((1-\alpha)/\rho)})^{1/(1-\alpha)} = vW \tag{3}$$

where

$$v = (E[(\hat{c}^\rho + \beta[(1 - \hat{c})\tilde{r}]^\rho]^{((1-\alpha)/\rho)})^{1/(1-\alpha)} \tag{4}$$

and

$$\tilde{r} = r_f + \hat{\gamma}\tilde{x}_m. \tag{5}$$

Note that  $\hat{\gamma}$  typically does depends on  $\rho$  as well as on the relative risk aversion measure  $\alpha$ .

Note also that, if there is no riskless asset, then

$$\hat{c} = \arg \max_c (E[(cW)^\rho + \beta[W(1 - c)\tilde{r}_m]^\rho]^{((1-\alpha)/\rho)})^{1/(1-\alpha)},$$

$$V(W) = \max_c (E[(cW)^\rho + \beta[W(1 - c)\tilde{r}_m]^\rho]^{((1-\alpha)/\rho)})^{1/(1-\alpha)} = vW$$

and  $v$  is given by (4) in which  $\tilde{r} = \tilde{r}_m$ .

When the elasticity of substitution,  $\sigma$ , equals 1 then

$$\rho = 0$$

and

$$V(W) = \max_{(c, \gamma)} (E[cW]^{(1-\alpha)/(1+\beta)} [W(1 - c)(r_f + \gamma\tilde{x}_m)]^{\beta(1-\alpha)/(1+\beta)})^{1/(1-\alpha)} = vW \tag{6}$$

where now

$$v = [E(\tilde{r}^{\beta(1-\alpha)/(1+\beta)})]^{1/(1-\alpha)} \hat{c}^{1/(1+\beta)} (1 - \hat{c})^{\beta/(1+\beta)}, \tag{7}$$

$$\hat{c} = \arg \max_c c^{1/(1+\beta)} (1 - c)^{\beta/(1+\beta)}, \tag{8}$$

$\tilde{r}$  is given by (5) and

$$\hat{\gamma} = \arg \max_{\gamma} [E((r_f + \gamma \tilde{x}_m)^{\beta(1-\alpha)/(1+\beta)})]^{1/(1-\alpha)}.$$

If there is no riskless asset,  $\hat{c}$  is again the solution to (8) and

$$V(W) = \max_c [E(cW)^{(1-\alpha)/(1+\beta)} [W(1-c)\tilde{r}_m]^{\beta(1-\alpha)/(1+\beta)}]^{1/(1-\alpha)} = vW$$

where now  $v$  is given by (7) with  $\tilde{r} = \tilde{r}_m$ .

### 2.3. A two-period problem: EZW, Kreps–Porteus

In this section, we illustrate the difference between our expected utility maximizing approach and that taken by EZW Kreps–Porteus. Following Epstein–Zin and Weil we assume that the preferences are CES and that relative risk aversion is constant and denoted by  $\alpha$ . As in the above discussion, we only consider the case  $\alpha \neq 1$ . In this case, when the elasticity of substitution,  $\sigma$ , is unequal to 1 and

$$\rho \neq 0,$$

the EZW Kreps–Porteus approach is to solve the problem

$$\max_{(c,\gamma)} [(cW)^\rho + \beta [E((W[1-c][r_f + \gamma \tilde{x}_m])^{1-\alpha})]^{\rho/(1-\alpha)}]^{1/\rho} = \max_{(c,\gamma)} [W(c^\rho + \beta[1-c]^\rho [E([r_f + \gamma \tilde{x}_m]^{1-\alpha})]^{\rho/(1-\alpha)})]^{1/\rho}. \tag{9}$$

In order to motivate this approach note that once  $c$  and  $\gamma$  have been chosen, second period consumption is random and equal to

$$W[1-c][r_f + \gamma \tilde{x}_m].$$

If the consumer exhibits constant relative risk aversion when making choices that affect second period consumption and his relative risk aversion measure is  $\alpha$ , then

$$[E((W[1-c][r_f + \gamma \tilde{x}_m])^{1-\alpha})]^{1/(1-\alpha)}$$

is the “certainty equivalent second period consumption level.” It is then assumed that the consumer chooses  $c$  and  $\gamma$  to maximize

$$U^0(C_1, C_2) = (C_1^\rho + \beta C_2^\rho)^{1/\rho} \tag{10}$$

where

$$C_1 = cW \tag{11}$$

and  $C_2$  is set equal to the certainty equivalent level of second period consumption; i.e.,

$$C_2 = [E((W[1-c][r_f + \gamma \tilde{x}_m])^{1-\alpha})]^{1/(1-\alpha)}. \tag{12}$$

Note that the maximand in (9) is indeed obtained when (11) and (12) are substituted in (10). Also note that

$$[E((W[1-c][r_f + \gamma \tilde{x}_m])^{1-\alpha})]^{1/(1-\alpha)} = W[1-c][E([r_f + \gamma \tilde{x}_m]^{1-\alpha})]^{1/(1-\alpha)}.$$

Thus the certainty equivalent second period consumption is simply savings

$$W[1-c]$$

multiplied by

$$[E([r_f + \gamma \tilde{x}_m]^{1-\alpha})]^{1/(1-\alpha)} \tag{13}$$

which is the certainty equivalent return associated with the random return

$$r_f + \gamma \tilde{x}_m.$$

The virtue of this approach is that its solution is easily obtained. Indeed, to determine

$$(\hat{c}, \hat{\gamma}) = \arg \max_{(c, \gamma)} [(c^\rho + \beta[1 - c]^\rho E([r_f + \gamma \tilde{x}_m]^{1-\alpha})^{\rho/(1-\alpha)})^{1/\rho}]$$

we first choose  $\gamma$  to maximize the certainty equivalent return (13). Thus,  $\gamma$  is simply chosen to be

$$\hat{\gamma} = \arg \max_{\gamma} [E[r_f + \gamma \tilde{x}_m]^{1-\alpha}]^{1/(1-\alpha)} = \arg \max_{\gamma} \left[ \frac{E[r_f + \gamma \tilde{x}_m]^{1-\alpha}}{1 - \alpha} \right]. \tag{14}$$

The random return  $\tilde{r}$  associated with the optimal portfolio choice  $\hat{\gamma}$  is given by (5) and the corresponding certainty equivalent return is

$$(E\tilde{r}^{1-\alpha})^{1/(1-\alpha)} = (E[r_f + \hat{\gamma} \tilde{x}_m]^{1-\alpha})^{1/(1-\alpha)}.$$

It is then also a simple matter to determine  $\hat{c}$  as

$$\hat{c} = \arg \max_{c \in [0, 1]} [(c^\rho + \beta[1 - c]^\rho (E\tilde{r}^{1-\alpha})^{\rho/(1-\alpha)})^{1/\rho}] = \frac{(\beta(E\tilde{r}^{1-\alpha})^{\rho/(1-\alpha)})^{1/(\rho-1)}}{(1 + (\beta(E\tilde{r}^{1-\alpha})^{\rho/(1-\alpha)})^{1/(\rho-1)})}. \tag{15}$$

In this approach

$$V(W) = \max_{(c, \gamma)} [(cW)^\rho + \beta E((W[1 - c][r_f + \gamma \tilde{x}_m]^{1-\alpha})^{\rho/(1-\alpha)})^{1/\rho}]$$

and

$$V(W) = vW \tag{16}$$

where

$$v = [(\hat{c}^\rho + \beta[1 - \hat{c}]^\rho (E\tilde{r}^{1-\alpha})^{\rho/(1-\alpha)})^{1/\rho}]. \tag{17}$$

Eqs. (16) and (17) are the EZW, Kreps–Porteus analogs of (3) and (4).

It is obvious from (14) that, unlike in the Kihlstrom–Mirman approach,  $\hat{\gamma}$  depends on  $\alpha$  but not  $\rho$ . It is this result that the portfolio choice is independent of  $\rho$  that underlies the observation made by Weil and Kocherlakota that this model cannot explain the equity premium any more effectively than the standard model in which

$$1 - \alpha = \rho.$$

When there is no riskless asset,  $\hat{c}$  and  $v$  are still given by (15) and (17) respectively, but, in this case,

$$\tilde{r} = \tilde{r}_m.$$

If we apply the EZW Kreps–Porteus approach to the case in which the elasticity of substitution,  $\sigma$ , equals 1 and

$$\rho = 0$$

the consumer-investor's problem is

$$\max_{(c, \gamma)} [\log(cW) + \beta \log(E((W[1 - c][r_f + \gamma \tilde{x}_m]^{1-\alpha})^{1/(1-\alpha)})].$$

The solution is

$$\begin{aligned} (\hat{c}, \hat{\gamma}) &= \arg \max_{(c, \gamma)} [\log c + \beta \log([1 - c] E([r_f + \gamma \tilde{x}_m]^{1-\alpha})^{1/(1-\alpha)})] \\ &= \arg \max_{(c, \gamma)} [c^{1/(1+\beta)} [1 - c]^{\beta/(1+\beta)} E([r_f + \gamma \tilde{x}_m]^{1-\alpha})^{\beta/(1+\beta)(1-\alpha)}]. \end{aligned}$$

Again  $\hat{c}$  is the solution to (8) and  $\hat{\gamma}$  is given by (14). In this case,

$$V(W) = \max_{(c, \gamma)} [c^{1/(1+\beta)} [1 - c]^{\beta/(1+\beta)} E([r_f + \gamma \tilde{x}_m]^{1-\alpha})^{\beta/(1+\beta)(1-\alpha)}]$$

and  $V(W)$  is given by (16) where now

$$v = [E\tilde{r}^{1-\alpha}]^{\beta/(1+\beta)(1-\alpha)} \hat{c}^{1/(1+\beta)} [1 - \hat{c}]^{\beta/(1+\beta)} \tag{18}$$

and  $\tilde{r}$  is given by (5). Eq. (18) is the EZW, Kreps–Porteus analog of (7).



When there is no riskless asset,  $\hat{c}$  is again the solution to (8) and  $v$  is given by (18) in which  $\bar{r} = \bar{r}_m$ .  
When there is no risky asset, then EZW, Kreps–Porteus and Kihlstrom–Mirman consumer-investors both choose

$$\hat{c} = \arg \max_{c \in [0,1]} [(cW)^\rho + \beta[W(1-c)r_f]^\rho]^{1/\rho} = \arg \max_{c \in [0,1]} [c^\rho + \beta[(1-c)r_f]^\rho]^{1/\rho}$$

when

$$\rho \neq 0.$$

When

$$\rho = 0$$

EZW, Kreps–Porteus and Kihlstrom–Mirman consumer-investors both choose

$$\hat{c} = \arg \max_{c \in [0,1]} [c^{1/(1+\beta)} [1-c]^{\beta/(1+\beta)} r_f^{\beta/(1+\beta)}]$$

which once again is the solution to (8).

### 3. The three-period case: the “consistent planning” approach with “forward looking” von Neumann–Morgenstern preferences

We now suppose that the consumer-investor lives for three periods and begins the first period with initial wealth,  $W_1$ . His consumption in period  $t$  is  $C_t$ . In period 3,

$$C_3 = W_3.$$

In periods  $t = 1$  and 2,

$$C_t = c_t W_t$$

where  $c_t$  is the fraction of period  $t$  wealth,  $W_t$ , consumed. So in periods  $t = 1$  and 2, the consumer-investor saves  $(1 - c_t)W_t$ , and can invest in a safe and a risky asset. Thus, in each of the first two periods, the consumer-investor must also choose the fraction,  $\gamma_t$ , of period  $t$  savings to invest in the risky asset. The safe asset return is  $r_f$ . The realized return on the risky asset is  $r_t$ . The random variable that denotes the risky asset return is  $\tilde{r}_t$ . We assume that  $\tilde{r}_1$  and  $\tilde{r}_2$  are i.i.d.. We again define the realized excess return on the risky asset as

$$x_t = r_t - r_f.$$

This is again the realization of the random variable

$$\tilde{x}_t = \tilde{r}_t - r_f.$$

Period  $t + 1$  wealth is then

$$W_{t+1} = W_t [1 - c_t] [r_f + \gamma_t \tilde{x}_t].$$

In discussing forward looking preferences we are going to assume that, in each period, the preferences are CES and exhibit CRRA risk aversion in the Kihlstrom–Mirman sense. Furthermore, the elasticity of substitution and the level of relative risk aversion are the same each period. Thus, in the first period, the consumer-investor maximizes

$$EU(C_1, \tilde{C}_2, \tilde{C}_3) = \frac{1}{1-\alpha} E[(C_1^\rho + \beta \tilde{C}_2^\rho + \beta^2 \tilde{C}_3^\rho)^{(1-\alpha)/\rho}]$$

where

$$C_1 = c_1 W_1,$$

$$C_2 = c_2 W_2 = c_2 W_1 [1 - c_1] [r_f + \gamma_1 x_1]$$

and

$$C_3 = W_2 [1 - c_2] [r_f + \gamma_2 x_2] = W_1 [1 - c_1] [r_f + \gamma_1 x_1] [1 - c_2] [r_f + \gamma_2 x_2].$$

In the second period, the consumer-investor maximizes

$$EU(C_2, \tilde{C}_3) = \frac{1}{1-\alpha} E[(C_2^\rho + \beta \tilde{C}_3^\rho)^{(1-\alpha)/\rho}].$$

In this situation, the need for “consistent planning” arises when

$$1 - \rho \neq \alpha$$

and when the consumer cannot commit to the choice of  $(c_2, \gamma_2)$  in period 1. To see why this is so, let us consider the case in which commitment is possible. In that case, the consumer’s first period choice would be

$$(\hat{c}_1, \hat{\gamma}_1, \hat{c}_2(\cdot), \hat{\gamma}_2(\cdot)) = \arg \max_{(c_1, \gamma_1, c_2(\cdot), \gamma_2(\cdot))} \left[ \frac{W_1^{1-\alpha}}{1-\alpha} \right] E[(c_1^\rho + \beta[1 - c_1]^\rho[r_f + \gamma_1 \tilde{x}_1]^\rho v(c_2(\tilde{x}_1), \gamma_2(\tilde{x}_1), \tilde{x}_2))^{(1-\alpha)/\rho}]$$

where

$$v(c_2, \gamma_2, x_2) = c_2^\rho + \beta[1 - c_2]^\rho[r_f + \gamma_2 x_2]^\rho.$$

Note that, for each  $x_1$ ,

$$\begin{aligned} (\hat{c}_2(x_1), \hat{\gamma}_2(x_1)) &= \arg \max_{(c_2, \gamma_2)} \left[ \frac{1}{1-\alpha} \right] E[(\hat{c}_1^\rho + \beta[1 - \hat{c}_1]^\rho[r_f + \hat{\gamma}_1 x_1]^\rho (c_2^\rho + \beta[1 - c_2]^\rho[r_f + \gamma_2 \tilde{x}_2]^\rho))^{(1-\alpha)/\rho}] \\ &= \arg \max_{(c_2, \gamma_2)} \left[ \frac{1}{1-\alpha} \right] E[(A(x_1)]^\rho + (c_2^\rho + \beta[1 - c_2]^\rho[r_f + \gamma_2 \tilde{x}_2]^\rho))^{(1-\alpha)/\rho}] \end{aligned}$$

where

$$A(x_1) = \frac{\hat{c}_1}{\beta^{1/\rho}[1 - \hat{c}_1][r_f + \hat{\gamma}_1 x_1]}.$$

When

$$1 - \rho \neq \alpha, \\ (\hat{c}_2(x_1), \hat{\gamma}_2(x_1))$$

varies with  $x_1$ . We will describe the nature of this variation in **Proposition 1** at the end of this section.

Now suppose that commitment is impossible. In that case, the forward looking von Neumann–Morgenstern consumer-investor is able, in period 2, to choose

$$\begin{aligned} (c_2^*, \gamma_2^*) &= \arg \max_{(c_2, \gamma_2)} \left[ \frac{W_2^{1-\alpha}}{1-\alpha} \right] E[(c_2^\rho + \beta[1 - c_2]^\rho[r_f + \gamma_2 \tilde{x}_2]^\rho)^{(1-\alpha)/\rho}] \\ &= \arg \max_{(c_2, \gamma_2)} \left[ \frac{1}{1-\alpha} \right] E[(c_2^\rho + \beta[1 - c_2]^\rho[r_f + \gamma_2 \tilde{x}_2]^\rho)^{(1-\alpha)/\rho}]. \end{aligned}$$

Since  $(c_2^*, \gamma_2^*)$  is independent of  $W_2$  it is also independent of  $x_1$  and we must have

$$(c_2^*, \gamma_2^*) \neq (\hat{c}_2(x_1), \hat{\gamma}_2(x_1)).$$

It should be noted that, when

$$1 - \rho = \alpha,$$

the usual dynamic programming argument implies that, for all  $x_1$ ,

$$(c_2^*, \gamma_2^*) = (\hat{c}_2(x_1), \hat{\gamma}_2(x_1)).$$

How should the forward looking von Neumann–Morgenstern consumer-investor make his choice in the first period if he knows that he will choose  $(c_2^*, \gamma_2^*)$  in the second period? As Strotz suggests we assume that he simply treats his “future misbehavior” as a constraint and makes a first period choice that is, in the game theoretic language of Peleg and Yaari, a best response to  $(c_2^*, \gamma_2^*)$ . The best response to  $(c_2^*, \gamma_2^*)$  is

$$\begin{aligned} (c_1^*, \gamma_1^*) &= \arg \max_{(c_1, \gamma_1)} W_1^{1-\alpha} \left[ \frac{1}{1-\alpha} \right] E[(c_1^\rho + \beta[1 - c_1]^\rho [r_f + \gamma_1 \tilde{x}_1]^\rho v(c_2^*, \gamma_2^*, \tilde{x}_2))^{((1-\alpha)/\rho)}] \\ &= \arg \max_{(c_1, \gamma_1)} \left[ \frac{1}{1-\alpha} \right] E[(c_1^\rho + \beta[1 - c_1]^\rho [r_f + \gamma_1 \tilde{x}_1]^\rho v(c_2^*, \gamma_2^*, \tilde{x}_2))^{((1-\alpha)/\rho)}]. \end{aligned}$$

It is natural to ask how the consumer-investor’s second period choice is affected by his ability to commit to a choice in the first period. The answer is provided by the following proposition.

**Proposition 1.** *Suppose that*

$$1 - \rho > (<) \alpha.$$

*In that case, for all  $x_1$ , the consumer-investor acts as if he were more (less) risk averse in the Kihlstrom–Mirman sense when he commits to*

$$(\hat{c}_2(x_1), \hat{\gamma}_2(x_1))$$

*in the first period than he does when he chooses*

$$(c_2^*, \gamma_2^*)$$

*in the second period. Also, in this case, increases in  $x_1$  cause the consumer-investor to act as if he were less (more) risk averse in the Kihlstrom–Mirman sense when he commits to*

$$(\hat{c}_2(x_1), \hat{\gamma}_2(x_1))$$

*in the first period.*

The proof of this proposition is in [Appendix 1](#).

#### 4. The infinite horizon case

We now suppose that the consumer-investor looks forward to an infinite lifetime. He begins period  $t$  with initial wealth,  $W_t$ . His consumption in period  $t$  is  $C_t$ . The fraction of  $W_t$  consumed in period  $t$  is  $c_t$ . In each period, savings, which equal  $(1 - c_t)W_t$ , can be invested in a safe and a risky asset. Thus, in each period, the consumer-investor must also choose the fraction,  $\gamma_t$ , of period  $t$  savings to invest in the risky asset. In each period, the safe asset return is  $r_f$ . The realized return on the risky asset is  $r_t$ . The random variable of which  $r_t$  is the realization is  $\tilde{r}_t$ . We assume that the  $\tilde{r}_t$ ’s are i.i.d.. We again define the realized excess return on the risky asset as

$$x_t = r_t - r_f.$$

This is again the realization of the random variable

$$\tilde{x}_t = \tilde{r}_t - r_f.$$

Period  $t + 1$  wealth is

$$W_{t+1} = W_t[1 - c_t][r_f + \gamma_t \tilde{x}_t]$$

and consumption in the period  $t$  is

$$C_t = c_t W_t.$$

##### 4.1. The “consistent planning” approach with “forward looking” von Neumann–Morgenstern preferences

In discussing, forward looking preferences in this infinite horizon setting, we are once again going to assume that, in each period, the preferences are CES and exhibit CRRA risk aversion in the Kihlstrom–Mirman sense. Furthermore, the elasticity of substitution and the level of relative risk aversion are the same each period.

Thus, in period  $t$ , the forward looking von Neumann–Morgenstern consumer-investor maximizes

$$EU(C_t, \{\tilde{C}_{t+\tau}\}_{\tau=1}^\infty) = \frac{1}{1-\alpha} E \left[ \left( C_t^\rho + \sum_{\tau=1}^\infty \beta^\tau \tilde{C}_{t+\tau}^\rho \right)^{(1-\alpha)/\rho} \right].$$

The risk neutral case is that in which

$$U(C_t, \{C_{t+\tau}\}_{\tau=1}^{\infty}) = \left( C_t^\rho + \sum_{\tau=1}^{\infty} \beta^\tau C_{t+\tau}^\rho \right)^{1/\rho}.$$

The Kihlstrom–Mirman relative risk aversion of

$$U(C_t, \{C_{t+\tau}\}_{\tau=1}^{\infty}) = \frac{1}{1-\alpha} \left( C_t^\rho + \sum_{\tau=1}^{\infty} \beta^\tau C_{t+\tau}^\rho \right)^{(1-\alpha)/\rho} \tag{19}$$

is  $\alpha$ . For all  $\alpha$ , the elasticity of substitution of (19) is

$$\sigma = \frac{1}{1-\rho}.$$

In this situation, the need for “consistent planning” again arises when

$$1 - \rho \neq \alpha$$

and when the consumer cannot commit to the choice of  $(c_{t+\tau}, \gamma_{t+\tau})$  in period  $\tau > 0$ . Thus, when the consumer chooses  $(c_t, \gamma_t)$  in period  $t$  he anticipates his future choices  $\{(c_{t+\tau}, \gamma_{t+\tau})\}_{\tau=1}^{\infty}$ . Because the consumer-investor faces the same problem in every period, the equilibrium is one in which his choice in every period is the same. Specifically, for all  $t$ ,  $(c_t, \gamma_t) = (\hat{c}, \hat{\gamma})$  where  $(\hat{c}, \hat{\gamma}) = (\hat{c}, \hat{\gamma})$  is a best response to the fact that for all,  $\tau > 0$ ,  $(c_{t+\tau}, \gamma_{t+\tau}) = (\hat{c}, \hat{\gamma})$ . Thus,

$$(\hat{c}, \hat{\gamma}) = \arg \max_{(c, \gamma)} (E[(c^\rho + \beta[1-c]^\rho [r_f + \gamma \bar{x}_t]^\rho v(\{\bar{x}_{t+\tau}\}_{\tau=1}^{\infty}, (\hat{c}, \hat{\gamma})))]^{(1-\alpha)/\rho})^{1/(1-\alpha)} \tag{20}$$

where, for each possible sample path of excess returns,  $\{x_{t+\tau}\}_{\tau=1}^{\infty}$ ,

$$v(\{x_{t+\tau}\}_{\tau=1}^{\infty}, (c, \gamma)) = c^\rho + \beta[1-c]^\rho [r_f + \gamma x_{t+1}]^\rho v(\{x_{t+1+\tau}\}_{\tau=1}^{\infty}, (c, \gamma)). \tag{21}$$

It is easy to check that when

$$v(\{x_{t+\tau}\}_{\tau=1}^{\infty}, (c, \gamma)) = c^\rho \left[ 1 + \sum_{\tau=1}^{\infty} \beta^\tau [1-c]^\rho \prod_{s=1}^{\tau} [r_f + \gamma x_{t+s}]^\rho \right]$$

(21) holds.

Since we are going to apply this approach to derive the equilibrium in the Lucas asset pricing model, it is useful to describe the consistent planning equilibrium for the case in which there is no riskless asset. In this case, the consumer simply chooses  $c_t$  in each period. Once again, the consumer-investor faces the same problem in every period, and the equilibrium is one in which his choice in every period is the same. Specifically, for all  $t$ ,  $c_t = \hat{c}$  where  $c_t = \hat{c}$  is a best response to the fact that for all,  $\tau > 0$ ,  $c_{t+\tau} = \hat{c}$ . Thus,

$$\hat{c} = \arg \max_c (E[(c^\rho + \beta[1-c]^\rho \bar{r}_t^\rho (v(\{\bar{r}_{t+\tau}\}_{\tau=1}^{\infty}, \hat{c})))]^{(1-\alpha)/\rho})^{1/(1-\alpha)}$$

where, for each sequence  $\{r_{t+\tau}\}_{\tau=1}^{\infty}$ ,

$$v(\{r_{t+\tau}\}_{\tau=1}^{\infty}, c) = c^\rho + \beta[1-c]^\rho r_{t+1}^\rho v(\{r_{t+1+\tau}\}_{\tau=1}^{\infty}, c). \tag{22}$$

It is again easy to check that when

$$v(\{r_{t+\tau}\}_{\tau=1}^{\infty}, c) = c^\rho \left[ 1 + \sum_{\tau=1}^{\infty} \beta^\tau [1-c]^\rho \prod_{s=1}^{\tau} r_{t+s}^\rho \right] \tag{23}$$

(22) holds.

Andrew Postlewaite has pointed out that when we follow Peleg and Yaari and interpret the equilibrium just described as an equilibrium of the Stackleberg game between the current consumer and the future consumer-investors, this equilibrium is not unique. The problem is that, in infinite games of this type, subgame perfect equilibria can exist in which the current consumer-investor anticipates punishment by the future consumer-investors if he does not deviate in particular ways from the equilibrium just described. Nevertheless, the equilibrium just described does seem to be the natural one to consider. In fact, Andreu Mas Colell has pointed out that this non-uniqueness problem arises even in the additively separable case. In that case, it has never been an obstacle to focusing on the outcome just described. We describe that case in the next section.

4.1.1. The additively separable case

As we have noted, the additively separable case is a special case of the model just described. This case arises in our approach when

$$1 - \alpha = \rho$$

which, of course, implies that

$$\sigma = \frac{1}{\alpha}.$$

In that case, the fact the  $\{\tilde{x}_{t+\tau}\}_{\tau=0}^{\infty}$  are i.i.d. implies that

$$\begin{aligned} E[(cW_t)^\rho + \beta W_t^\rho [1 - c]^\rho [r_f + \gamma \tilde{x}_t]^\rho v(\{\tilde{x}_{t+\tau}\}_{\tau=1}^{\infty}, (\hat{c}, \hat{\gamma}))^{(1-\alpha)/\rho}] \\ = [(cW_t)^{1-\alpha} + \beta W_t^{1-\alpha} [1 - c]^{1-\alpha} E[r_f + \gamma \tilde{x}_t]^{1-\alpha} E(v(\{\tilde{x}_{t+\tau}\}_{\tau=1}^{\infty}, (\hat{c}, \hat{\gamma})))] \end{aligned}$$

Thus, (20) reduces to

$$\begin{aligned} (\hat{c}, \hat{\gamma}) &= \arg \max_{(c, \gamma)} [c^{1-\alpha} + \beta v^{1-\alpha} [1 - c]^{1-\alpha} E[r_f + \gamma \tilde{x}_t]^{1-\alpha}]^{1/(1-\alpha)} \\ &= \arg \max_{(c, \gamma)} \left[ \frac{1}{1 - \alpha} \right] [c^{1-\alpha} + \beta v^{1-\alpha} [1 - c]^{1-\alpha} E[r_f + \gamma \tilde{x}_t]^{1-\alpha}] \end{aligned} \tag{24}$$

where

$$v^{1-\alpha} = E(v(\{\tilde{x}_{t+\tau}\}_{\tau=1}^{\infty}, (\hat{c}, \hat{\gamma}))) = \hat{c}^{1-\alpha} \left[ 1 + \sum_{\tau=1}^{\infty} \beta^\tau [1 - \hat{c}]^{(1-\alpha)\tau} (E\tilde{r}^{1-\alpha})^\tau \right] = \hat{c}^{1-\alpha} \left[ \frac{1}{1 - \beta [1 - \hat{c}]^{(1-\alpha)} E\tilde{r}^{1-\alpha}} \right], \tag{25}$$

$$\tilde{r} = r_f + \hat{\gamma} \tilde{x}_t$$

and

$$\hat{\gamma} = \arg \max_{\gamma} \left[ \frac{1}{1 - \alpha} \right] E[r_f + \gamma \tilde{x}_t]^{1-\alpha}.$$

The solution is

$$\hat{c} = \frac{1}{1 + (\beta v^{1-\alpha} E\tilde{r}^{1-\alpha})^{1/\alpha}}. \tag{26}$$

Substituting (26) in (25) and solving for  $v$  the result is

$$v^{1-\alpha} = [1 - (\beta E\tilde{r}^{1-\alpha})^{1/\alpha}]^{-\alpha}. \tag{27}$$

Substituting this in (26) yields

$$\hat{c} = [1 - (\beta E\tilde{r}^{1-\alpha})^{1/\alpha}]. \tag{28}$$

Note that  $\hat{c}$  is positive only if

$$\frac{1}{\beta} > E\tilde{r}^{1-\alpha}.$$

When  $v$  and  $\hat{c}$  satisfy (27) and (28)

$$v^{1-\alpha} = [\hat{c}^{1-\alpha} + \beta v^{1-\alpha} [1 - \hat{c}]^{1-\alpha} E\tilde{r}^{1-\alpha}]$$

so that

$$v = \max_{(c, \gamma)} [c^{1-\alpha} + \beta v^{1-\alpha} [1 - c]^{1-\alpha} E[r_f + \gamma \tilde{x}_t]^{1-\alpha}]^{1/(1-\alpha)}. \tag{29}$$

4.2. Epstein–Zin and Weil

In the current setting, the EZW approach yields an equilibrium in which the optimal choice for  $(c_t, \gamma_t)$  in each period,  $t$ , is

$$(\hat{c}, \hat{\gamma}) = \arg \max_{(c, \gamma)} [((cW)^\rho + \beta [E\{V(W[1 - c][r + \gamma \tilde{x}_t])\}]^{1-\alpha})^{\rho/(1-\alpha)}]^{1/\rho}$$

where  $V(\cdot)$  solves the functional equation

$$V(W) = \max_{(c,\gamma)} [(cW)^\rho + \beta E([V(W[1-c][r_f + \gamma\tilde{x}_t])^{1-\alpha}]^{\rho/(1-\alpha)})^{1/\rho}]. \tag{30}$$

A well-known advantage of the EZW approach is its tractability. First note that the solution to the functional equation (30) is

$$V(W) = vW. \tag{31}$$

Using (31) the problem of solving the functional equation (30) reduces to finding  $v$  that solves the equation

$$v = \max_{(c,\gamma)} [(c^\rho + \beta v^\rho [1-c]^\rho (E[r_f + \gamma\tilde{x}_t]^{1-\alpha})^{\rho/(1-\alpha)})^{1/\rho}] \tag{32}$$

which we now proceed to do. In addition, (31) implies that

$$(\hat{c}, \hat{\gamma}) = \arg \max_{(c,\gamma)} [(c^\rho + \beta v^\rho [1-c]^\rho (E[r_f + \gamma\tilde{x}_t]^{1-\alpha})^{\rho/(1-\alpha)})^{1/\rho}]. \tag{33}$$

The solution can be found by noting that

$$\hat{\gamma} = \arg \max_{\gamma} [E[r_f + \gamma\tilde{x}_t]^{1-\alpha}]^{1/(1-\alpha)} = \arg \max_{\gamma} \left[ \frac{E[r_f + \gamma\tilde{x}_t]^{1-\alpha}}{1-\alpha} \right], \tag{34}$$

and

$$\hat{c} = \arg \max_c [(c^\rho + \beta v^\rho [1-c]^\rho (E\tilde{r}^{1-\alpha})^{\rho/(1-\alpha)})^{1/\rho}] = \frac{1}{1 + (\beta v^\rho (E\tilde{r}^{1-\alpha})^{\rho/(1-\alpha)})^{1/(1-\rho)}}, \tag{35}$$

where

$$\tilde{r} = r_f + \hat{\gamma}\tilde{x}_t. \tag{36}$$

It is clear from (34) that  $\hat{\gamma}$  depends only on the relative risk aversion measure  $\alpha$  and is independent of  $\rho$ . As mentioned in Section 1, this is a feature of the EZW optimal portfolio that has been noted and commented on by a number of authors.

Using the expression (35) that determines  $\hat{c}$  we can solve (32) for  $v$  to get

$$v = [1 - (\beta (E\tilde{r}^{1-\alpha})^{\rho/(1-\alpha)})^{1/(1-\rho)}]^{(\rho-1)/\rho} \tag{37}$$

which is positive if and only if

$$\frac{1}{\beta} > (E\tilde{r}^{1-\alpha})^{\rho/(1-\alpha)}.$$

Thus,  $V(W)$  becomes

$$V(W) = [1 - (\beta (E\tilde{r}^{1-\alpha})^{\rho/(1-\alpha)})^{1/(1-\rho)}]^{(\rho-1)/\rho} W.$$

Also the expression for  $\hat{c}$  simplifies to

$$\hat{c} = [1 - (\beta (E\tilde{r}^{1-\alpha})^{\rho/(1-\alpha)})^{1/(1-\rho)}]. \tag{38}$$

In this approach,  $\alpha$  is interpreted as the measure of relative risk aversion. The elasticity of substitution is of course

$$\sigma = \frac{1}{1-\rho}.$$

When the EZW approach is applied to derive the equilibrium in the Lucas asset pricing model, there is no riskless asset. For this case, in each period, the consumer simply chooses  $c_t = \hat{c}$  where  $\hat{c}$  satisfies (38) and

$$\tilde{r} = \tilde{r}_t. \tag{39}$$

In this case, (32) is replaced by

$$v = \max_c [(c^\rho + \beta v^\rho [1-c]^\rho (E\tilde{r}_t^{1-\alpha})^{\rho/(1-\alpha)})^{1/\rho}].$$

In this case,  $v$  is again given by (37) where now  $\tilde{r}$  is as in (39).

4.2.1. *The additively separable case*

This case arises when

$$1 - \alpha = \rho$$

which, of course, implies that

$$\sigma = \frac{1}{\alpha}.$$

In this case, (37) and (38) reduce to (27) and (28), respectively. Also, (32) and (33) reduce to (29) and (24), respectively.

**5. The infinite period Lucas asset pricing equilibrium**

Assume that the Lucas tree dividends are  $\tilde{s}_t$  and that

$$\tilde{s}_{t+1} = \tilde{g}_t s_t.$$

We assume that the dividend growth rates  $\tilde{g}_t$  are i.i.d. and that

$$Pr(\tilde{g}_t > 0) = 1.$$

Let  $\tilde{g}$  be a random variable with the same distribution as  $\tilde{g}_t$  for all  $t$ . Let  $P(s_t)$  be the period  $t$  price of the tree. Then the return on savings invested at time  $t$  is

$$r(s_{t+1}, s_t) = \frac{s_{t+1} + P(s_{t+1})}{P(s_t)}. \tag{40}$$

In equilibrium, wealth in period  $t$  is

$$W_t(s_t) = s_t + P(s_t),$$

consumption in period  $t$  is  $s_t$  and savings in period  $t$  is  $P(s_t)$ .

In what follows, the assumption that the growth rates  $\tilde{g}_t$  are i.i.d. will make it possible to describe an equilibrium in which  $c_t$  is the same in every period  $t$ . When, for all  $t$ ,  $c_t = \hat{c}$ , then

$$\hat{c} = \frac{s_t}{s_t + P(s_t)}$$

and

$$1 - \hat{c} = \frac{P(s_t)}{s_t + P(s_t)}.$$

This implies that the price dividend ratio is

$$\frac{P(s_t)}{s_t} = \frac{1 - \hat{c}}{\hat{c}}$$

and that

$$P(s_t) = \frac{1 - \hat{c}}{\hat{c}} s_t. \tag{41}$$

Substituting (41) in (40) we get that

$$r_t = r(s_{t+1}, s_t) = \frac{1}{1 - \hat{c}} \frac{s_{t+1}}{s_t} = \frac{1}{1 - \hat{c}} \tilde{g}_t. \tag{42}$$

Note that (42) implies that the expected return on Lucas tree each period is

$$E\tilde{r} = \frac{E\tilde{g}}{1 - \hat{c}}. \tag{43}$$

Also (41) can be rewritten as

$$P(s_t) = \frac{1}{(1/(1 - \hat{c})) - 1} s_t. \tag{44}$$

Together with (43), (44) implies

$$P(s_t) = \frac{E\tilde{g}}{E\tilde{r} - E\tilde{g}} s_t. \tag{45}$$

Thus the equilibrium value of the Lucas tree at time  $t$  is the present value of a dividend stream that starts at  $s_t$  and grows at the expected rate  $E\tilde{g}$ . The discount rate used to compute the present value is the expected return on the Lucas tree.

It should be pointed out that the equilibrium results just described are all a consequence of the fact that the growth rates  $\tilde{g}_t$  are i.i.d. and that, for all  $t$ ,  $c_t = \hat{c}$ . They do not depend on any assumption about preferences. This will imply that these results apply in the consistent planning approach as well as in the Epstein–Zin and Weil approaches.

5.1. The “consistent planning” approach with “forward looking” von Neumann–Morgenstern preferences

In this case, for all  $t$ ,  $c_t = \hat{c}$  where

$$\hat{c} = \arg \max_c (E[(c^\rho + \beta[1 - c]^\rho \tilde{r}_t^\rho (v(\{\tilde{r}_{t+\tau}\}_{\tau=1}^\infty, \hat{c}))^{(1-\alpha/\rho)}])^{1/(1-\alpha)} \tag{46}$$

and where, for each sequence,  $\{r_{t+\tau}\}_{\tau=1}^\infty$ ,

$$v(\{r_{t+\tau}\}_{\tau=1}^\infty, \hat{c}) = \hat{c}^\rho \left[ 1 + \sum_{\tau=1}^\infty \beta^\tau [1 - \hat{c}]^{\rho\tau} \prod_{s=1}^\tau r_{t+s}^\rho \right]. \tag{47}$$

Using (42), (47) reduces to

$$r_t^\rho v(\{r_{t+\tau}\}_{\tau=1}^\infty, \hat{c}) = r_t^\rho v\left(\left\{\frac{1}{1-\hat{c}}g_{t+\tau}\right\}_{\tau=1}^\infty, \hat{c}\right) = g_t^\rho \left[\frac{\hat{c}}{1-\hat{c}}\right]^\rho \theta(\{g_{t+\tau}\}_{\tau=1}^\infty) \tag{48}$$

where

$$\theta(\{g_{t+\tau}\}_{\tau=1}^\infty) = \left[ 1 + \sum_{\tau=1}^\infty \beta^\tau \prod_{s=1}^\tau g_{t+s}^\rho \right]. \tag{49}$$

Using (48), the maximand in (46) becomes

$$E \left[ \left( c^\rho + \beta[1 - c]^\rho \left[ \frac{\hat{c}}{1-\hat{c}} \right]^\rho \tilde{g}_t^\rho \theta(\{\tilde{g}_{t+\tau}\}_{\tau=1}^\infty) \right)^{(1-\alpha)/\rho} \right] \tag{50}$$

which is shown to be finite in the following proposition.

**Proposition 2.** When  $\hat{c} \in (0, 1)$  and  $c \in (0, 1)$  the expectation (50) is finite and

$$\theta(\{g_{t+\tau}\}_{\tau=1}^\infty)$$

is finite almost surely if

$$\rho < 1 - \alpha < 0 \tag{51}$$

and

$$\beta E \tilde{g}^\rho < 1. \tag{52}$$

The proof of this proposition is in Appendix 2. The parameter restriction imposed in condition (51) implies that the relative risk aversion  $\alpha$  exceeds one and that the elasticity of substitution is below the level it takes on in the additively separable case. Previous empirical literature suggests that the case in which the relative risk aversion exceeds one is clearly of most interest. As noted in Section 1, and as demonstrated below in Proposition 4, the equity premium that results from the equilibrium obtained with the forward looking von Neumann–Morgenstern preferences exceeds that obtained from the additively separable model only when the elasticity of substitution falls below the level it is restricted to take on in the additively separable case.

The first-order condition satisfied at  $\hat{c}$  is

$$E \left[ \left( 1 + \beta \tilde{g}_t^\rho \tilde{\theta} \right)^{((1-\alpha)/\rho)-1} \left( 1 - \beta \left[ \frac{\hat{c}}{1-\hat{c}} \right] \tilde{g}_t^\rho \tilde{\theta} \right) \right] = 0 \tag{53}$$

where

$$\tilde{\theta} = \theta(\{\tilde{g}_{t+\tau}\}_{\tau=1}^\infty).$$

Solving (53) we get

$$\frac{P(s_t)}{s_t} = \left[ \frac{1-\hat{c}}{\hat{c}} \right] = \frac{\beta E[(1 + \beta \tilde{g}_t^\rho \tilde{\theta})^{((1-\alpha)/\rho)-1} \tilde{g}_t^\rho \tilde{\theta}]}{E(1 + \beta \tilde{g}_t^\rho \tilde{\theta})^{((1-\alpha)/\rho)-1}}. \tag{54}$$



which implies that

$$\hat{c} = \frac{E(1 + \beta \tilde{g}_t^\rho \tilde{\theta})^{((1-\alpha)/\rho)-1}}{E(1 + \beta \tilde{g}_t^\rho \tilde{\theta})^{((1-\alpha)/\rho)}} \tag{55}$$

and

$$1 - \hat{c} = \frac{\beta E[(1 + \beta \tilde{g}_t^\rho \tilde{\theta})^{((1-\alpha)/\rho)-1} \tilde{g}_t^\rho \tilde{\theta}]}{E(1 + \beta \tilde{g}_t^\rho \tilde{\theta})^{((1-\alpha)/\rho)}}. \tag{56}$$

To begin to interpret these results and to use them to derive an “equity premium,” suppose that a risk-free asset is introduced but that it is in zero net supply. Then  $r_f$ , the return on the risk-free asset will have to be such that

$$1 = \hat{\gamma} = \arg \max_{\gamma} [E[(\hat{c}^\rho + \beta(1 - \hat{c})[r_f + \gamma(\tilde{r}_t - r_f)]]^\rho (v(\{\tilde{r}_{t+\tau}\}_{\tau=1}^\infty, \hat{c}))^{(1-\alpha)/\rho}]^{1/(1-\alpha)}].$$

The first-order condition satisfied at  $\hat{\gamma} = 1$  implies that we must have

$$\frac{1}{r_f} = \frac{\beta \hat{c} E[(1 + \beta \tilde{g}_t^\rho \tilde{\theta})^{((1-\alpha)/\rho)-1} \tilde{g}_t^{\rho-1} \tilde{\theta}]}{E(1 + \beta \tilde{g}_t^\rho \tilde{\theta})^{((1-\alpha)/\rho)-1}} = \frac{\beta E[(1 + \beta \tilde{g}_t^\rho \tilde{\theta})^{((1-\alpha)/\rho)-1} \tilde{g}_t^{\rho-1} \tilde{\theta}]}{E(1 + \beta \tilde{g}_t^\rho \tilde{\theta})^{(1-\alpha)/\rho}} \tag{57}$$

In equilibrium the risk-free rate  $r_f$  equals the marginal rate of substitution (MRS) that measures the additional growth required to replace a sure dollar of consumption in period  $t$ . With the forward looking von Neumann–Morgenstern preferences assumed here, this MRS is measured by the inverse of the ratio in (57).

Note that (56) and (57) combine to imply that

$$1 - \hat{c} = \frac{1}{r_f} E^* \tilde{g}_t \tag{58}$$

where

$$E^* \tilde{g}_t = \frac{E[(1 + \beta \tilde{g}_t^\rho \tilde{\theta})^{((1-\alpha)/\rho)-1} \tilde{g}_t^\rho \tilde{\theta}]}{E[(1 + \beta \tilde{g}_t^\rho \tilde{\theta})^{((1-\alpha)/\rho)-1} \tilde{g}_t^{\rho-1} \tilde{\theta}]} \tag{59}$$

We can interpret  $E^* \tilde{g}_t$  as a “risk-neutral” expected growth rate of consumption and rewrite (58) as asserting that

$$r_f = \frac{E^* \tilde{g}_t}{(1 - \hat{c})} = E^* \tilde{r}_t \tag{60}$$

where  $E^* \tilde{r}_t$  is the “risk neutral” expected return paid by the Lucas tree.

**Remark 3.** In order to interpret (60) and justify the terminology “risk-neutral expectation” just used let

$$f(g)$$

be the probability density of  $\tilde{g}$  and note that

$$f^*(\{g_{t+\tau}\}_{\tau=0}^\infty) = \frac{(1 + \beta g_t^\rho \theta(\{g_{t+\tau}\}_{\tau=1}^\infty))^{((1-\alpha)/\rho)-1} g_t^{\rho-1} \theta(\{g_{t+\tau}\}_{\tau=1}^\infty) \prod_{\tau=0}^\infty f(g_{t+\tau})}{E[(1 + \beta \tilde{g}_t^\rho \theta(\{\tilde{g}_{t+\tau}\}_{\tau=1}^\infty))^{((1-\alpha)/\rho)-1} \tilde{g}_t^{\rho-1} \theta(\{\tilde{g}_{t+\tau}\}_{\tau=1}^\infty)]} \tag{61}$$

can also be regarded as a probability density of  $\{\tilde{g}_{t+\tau}\}_{\tau=0}^\infty$ . Observe that when  $f^*$  is used to compute the expected growth rate and the expected return paid by the Lucas tree we get  $E^* \tilde{g}_t$  and  $E^* \tilde{r}_t$ , respectively. In asset pricing models with risk neutral investors who invest in a safe and risky asset, the equation

$$r_f = E \tilde{r}_t \tag{62}$$

holds as an equilibrium condition. In (62) the expectation of  $\tilde{r}_t$ , the return on the risky asset, is computed using the true density of that return. Equation (60) is an analog of the “risk-neutral” equilibrium condition (62). In Eq. (60),  $f^*$  rather than the true density is used to compute the expectation of  $\tilde{r}_t$  and (60) holds even if the consumer-investor is not risk neutral. Because of this,  $f^*$  is commonly referred to as the “risk-neutral” density and  $E^* \tilde{g}_t$  and  $E^* \tilde{r}_t$  are referred to as “risk-neutral” expected values.

We can also interpret

$$f^*(\{g_{t+\tau}\}_{\tau=0}^\infty)$$

as the price of an Arrow Debreu contingent claim that pays one dollar when

$$\{\tilde{g}_{t+\tau}\}_{\tau=0}^{\infty} = \{g_{t+\tau}\}_{\tau=0}^{\infty}.$$

When defined in this way, the Arrow Debreu contingent claims prices are normalized to integrate to one. Using this interpretation  $E^*\tilde{g}_t$  is the value, computed using the Arrow Debreu contingent claims prices, of a portfolio that pays  $g_t$  at time  $t$  when

$$\{\tilde{g}_{t+\tau}\}_{\tau=0}^{\infty} = \{g_{t+\tau}\}_{\tau=0}^{\infty}.$$

Similarly,  $E_t^*\tilde{r}_t$  is the value, computed using the Arrow Debreu contingent claims prices, of the returns paid at time  $t$  by an investment in the Lucas tree. Eq. (60) asserts that the value of the returns paid by an investment in the Lucas tree equals the value of the sure return paid by an investment in the riskless asset, when the value of both of these returns are computed using the Arrow Debreu contingent claims prices.

Because

$$(1 + \beta g^{\rho} \theta)^{((1-\alpha)/\rho)-1} g^{\rho-1} \theta$$

is a decreasing function of  $g$ , we can easily show that

$$E^*\tilde{g} < E\tilde{g}. \tag{63}$$

Combining (43), (60) and (63) we have

$$r_f = E^*\tilde{r} = \frac{E^*\tilde{g}}{(1-\hat{c})} < \frac{E\tilde{g}}{(1-\hat{c})} = r_f \frac{E\tilde{g}}{E^*\tilde{g}} = E\tilde{r}. \tag{64}$$

The risk premium on the risky asset represented by the Lucas tree is

$$\frac{E\tilde{r}}{r_f} = \frac{E\tilde{g}}{E^*\tilde{g}}. \tag{65}$$

This is our main result. It implies that the equity risk premium is determined by

$$E^*\tilde{g} = \frac{E[(1 + \beta \tilde{g}^{\rho} \tilde{\theta})^{((1-\alpha)/\rho)-1} \tilde{g}^{\rho} \tilde{\theta}]}{E[(1 + \beta \tilde{g}^{\rho} \tilde{\theta})^{((1-\alpha)/\rho)-1} \tilde{g}^{\rho-1} \tilde{\theta}]} \tag{66}$$

which, in general, depends on both the elasticity of substitution

$$\sigma = \frac{1}{1-\rho}$$

and the risk aversion  $\alpha$ .

Finally, note that (60) and (44) combine to imply that

$$P(s_t) = \frac{E^*\tilde{g}}{E^*\tilde{r} - E^*\tilde{g}} s_t = \frac{E^*\tilde{g}}{r_f - E^*\tilde{g}} s_t.$$

Thus the equilibrium value of the Lucas tree at time  $t$  can also be computed as the present value of a dividend stream that starts at  $s_t$  and grows at the “risk neutral” expected rate  $E^*\tilde{g}$ . In this case, the discount rate used to compute the present value is the “risk neutral” expected return on the Lucas tree which equals the riskless rate.

### 5.1.1. The additively separable case

Note that when we are in the additively separable case,

$$\alpha = 1 - \rho$$

and (66) and (57) reduce to

$$E^*\tilde{g} = \frac{E\tilde{g}^{1-\alpha}}{E\tilde{g}^{-\alpha}} \tag{67}$$

and

$$r_f = \frac{1}{\beta E\tilde{g}_t^{-\alpha}}. \tag{68}$$

Substituting (42) in (28) we get

$$\hat{c} = 1 - \beta E\tilde{g}^{1-\alpha}.$$

The same result is obtained by substituting

$$E\tilde{\theta} = \left[ 1 + \sum_{\tau=1}^{\infty} (\beta E\tilde{g}^{1-\alpha})^{\tau} \right] = \frac{1}{1 - \beta E\tilde{g}^{1-\alpha}}$$

in

$$\hat{c} = \frac{1}{1 + \beta \tilde{g}_t^{1-\alpha} E\tilde{\theta}}$$

which is (55) in the additively separable case.

5.1.2. *The equity premium in the non-additively separable case*

Since the equity premium in (65) rises as  $E^*\tilde{g}$  falls, it is natural to ask when

$$\frac{E[(1 + \beta \tilde{g}^{\rho} \tilde{\theta})^{((1-\alpha)/\rho)-1} \tilde{g}^{\rho} \tilde{\theta}]}{E[(1 + \beta \tilde{g}^{\rho} \tilde{\theta})^{((1-\alpha)/\rho)-1} \tilde{g}^{\rho-1} \tilde{\theta}]} < \frac{E\tilde{g}^{1-\alpha}}{E\tilde{g}^{-\alpha}}. \tag{69}$$

The left-hand side of (69) is  $E^*\tilde{g}$  in our approach when

$$\alpha \neq 1 - \rho$$

and the right-hand side of (69) is  $E^*\tilde{g}$  in the additively separable case that arises as a special case of our approach when

$$\alpha = 1 - \rho.$$

Because the equity premium in (65) is inversely related to  $E^*\tilde{g}$ , (69) asserts that the equity premium implied by our approach exceeds the equity premium obtained from the additively separable case. Kocherlakota has pointed out that the following proposition is true.

**Proposition 4.** (Kocherlakota) *Inequality (69) holds when and only when*

$$\alpha < 1 - \rho. \tag{70}$$

The proof is given in Appendix 3.

Proposition 2 demonstrated that when conditions (70) and (52) hold expected utility is finite. It is perhaps interesting to note that (70) is also the condition under which the consumer-investor acts as if he were more risk averse in the Kihlstrom–Mirman sense when he commits to future consumption–portfolio plans than he does when he makes his consumption–portfolio choice in the future.

In the Epstein–Zin approach (70) implies that late resolution of uncertainty is preferred to early resolution of uncertainty. It must be emphasized, however, that unless

$$\alpha = 1 - \rho.$$

the Epstein–Zin risk-aversion measure has a different interpretation than the risk-aversion measure of our approach.

5.1.3. *The effect of risk aversion on savings in the non-additively separable case*

In the analysis of the two-period consumption savings model without a riskless asset in Kihlstrom and Mirman (1974) savings were shown to increase (decrease) with risk aversion if the elasticity of substitution is less (greater) than 1. Essentially the same result was obtained by Diamond and Stiglitz (1974). The proposition of this section obtains an analogous result for the equilibrium savings level in the Lucas asset pricing model just described. We demonstrate that  $1 - \hat{c}$  is a decreasing (increasing) function of the risk aversion parameter  $\alpha$  when the elasticity of substitution is greater (less) than one. Because of (42) the equilibrium return will rise for each possible level of growth if saving decreases. Thus, an increase in the risk aversion parameter will raise the equilibrium return for each possible level of growth if the elasticity of substitution is greater (less) than 1.

Let  $\hat{c}(\alpha)$  be defined as the solution to

$$E \left[ (1 + \beta \tilde{g}^{\rho} \tilde{\theta})^{((1-\alpha)/\rho)-1} \left( 1 - \beta \left[ \frac{\hat{c}(\alpha)}{1 - \hat{c}(\alpha)} \right] \tilde{g}^{\rho} \tilde{\theta} \right) \right] = 0$$

which is the first-order condition (53).

**Proposition 5.**  $1 - \hat{c}(\alpha)$  is a decreasing (increasing) function of  $\alpha$  if  $\sigma(\rho) = 1/(1 - \rho) > (<) 1$ .

The proof is given in Appendix 4.

**Corollary 6.** For each  $g$ , the equilibrium return

$$r = \frac{g}{1 - \hat{c}(\alpha)}$$

is an increasing (decreasing) function of  $\alpha$  if  $\sigma(\rho) = (1/1 - \rho) > (<) 1$ . This of course implies that the equilibrium expected return

$$E\tilde{r} = \frac{E\tilde{g}}{1 - \hat{c}(\alpha)}$$

is an increasing (decreasing) function of  $\alpha$  if  $\sigma(\rho) = 1/(1 - \rho) > (<) 1$ .

5.1.4. The effect of risk aversion on  $E^*g$  and the equity premium?

We would expect an increase in the risk aversion measure  $\alpha$  to result in a decrease in the risk-neutral expected growth rate:

$$E^*\tilde{g}$$

and an increase in the equity premium

$$\frac{E\tilde{g}}{E^*\tilde{g}}$$

As the next proposition demonstrates this turns out to be true under some conditions on the parameter values.

Let

$$E_1^*\tilde{g} = \frac{E[(1 + \beta\tilde{g}^\rho\tilde{\theta})^{((1-\alpha_1)/\rho)-1}(\tilde{g}^\rho\tilde{\theta})]}{E[(1 + \beta\tilde{g}^\rho\tilde{\theta})^{((1-\alpha_1)/\rho)-1}(\tilde{g}^{\rho-1}\tilde{\theta})]}$$

**Proposition 7.** Suppose that  $\rho < 0$  and

$$1 - \alpha_2 < \rho$$

or that  $\rho > 0$  and

$$1 - \alpha_2 > \rho.$$

Then

$$\alpha_2 < \alpha_1$$

implies

$$E_2^*\tilde{g} > E_1^*\tilde{g}. \tag{71}$$

The proof is in Appendix 5. Note that the sufficient conditions in the proposition are consistent with (70) only when  $\alpha_2 < 1$ .

5.2. Epstein–Zin and Weil

In this case, when a risk-free asset is introduced but is in zero net supply, the return on the risk-free asset,  $r_f$ , must be such that

$$1 = \hat{\gamma} = \arg \max_{\gamma} [E[r_f + \gamma\tilde{x}_t]^{1-\alpha}]^{1/(1-\alpha)}.$$

The first-order condition satisfied at  $1 = \hat{\gamma}$  implies that

$$r_f = E^*\tilde{r}_t \tag{72}$$

where

$$E^*\tilde{r}_t = \frac{E\tilde{r}_t^{1-\alpha}}{E\tilde{r}_t^{-\alpha}} \tag{73}$$

is now the “risk neutral” expected value of  $\tilde{r}_t$ . Substituting (42) in (73) yields

$$E^*\tilde{r}_t = \frac{E^*\tilde{g}_t}{(1 - \hat{c})}. \tag{74}$$

where now

$$E^* \tilde{g}_t = \frac{E \tilde{g}_t^{1-\alpha}}{E \tilde{g}_t^{-\alpha}} \tag{75}$$

can be interpreted as a risk-neutral expected growth rate. Since

$$g_t^{-\alpha}$$

is a decreasing function of  $g_t$ , we again have

$$E^* \tilde{g}_t < E \tilde{g}_t.$$

Thus, (72), (74) and (43) imply

$$r_f = E^* \tilde{r}_t = \frac{E^* \tilde{g}_t}{1 - \hat{c}} < \frac{E \tilde{g}_t}{1 - \hat{c}} = E \tilde{r}_t.$$

Combining (74) and (43) we obtain the expression

$$\frac{E \tilde{r}_t}{r_f} = \frac{E \tilde{g}_t}{E^* \tilde{g}_t} \tag{76}$$

for the equity risk premium. Note that the expression (75) for  $E^* \tilde{g}_t$  is the same as in the additively separable case. This expression also implies that the EZW equity risk premium in (76) is independent of  $\rho$ . This is the reason that Weil asserted that “If the dividend growth process is i.i.d., the risk-premium, when appropriately defined, is independent of the intertemporal elasticity of substitution, and thus is the same whether or not the time-additive, expected utility restriction is imposed.”

The solution for  $\hat{c}$  obtained in (38) implies that

$$1 - \hat{c} = (\beta(E \tilde{r}_t^{1-\alpha})^{\rho/(1-\alpha)})^{1/(1-\rho)}. \tag{77}$$

Substituting (42) in (77) yields

$$1 - \hat{c} = \beta(E \tilde{g}_t^{1-\alpha})^{\rho/(1-\alpha)}.$$

When this expression, (74) and (75) are substituted in (72) the result is

$$r_f = \frac{(E \tilde{g}_t^{1-\alpha})^{1-(\rho/(1-\alpha))}}{\beta E \tilde{g}_t^{-\alpha}}.$$

This expression for the risk-free rate differs from the expression (68) obtained for the additively separable case except when that case arises because

$$\rho = 1 - \alpha.$$

## 6. Summary

In this paper we have proposed a dynamic consumption-savings-portfolio choice model in which the consumer-investor maximizes the expected utility of a non-additively separable utility function of current and future consumption. The non-additive separability coupled with the fact that the consumer-investor’s expected utility is independent of past consumption implies that his choices are not dynamically consistent. If he could commit himself to future consumption plans he would make plans that he would not choose to carry out in the future if he were then free of past commitments. In the face of this dynamic inconsistency we follow the “consistent planning” approach of Strotz (1956) and assume that, when making his current choice, the consumer-investor will “take account of future disobedience.” This implies that he will choose a consumption-portfolio plan for the future that is, as Strotz asserted, “the best plan among those he will actually follow.” Our approach can also be interpreted from the perspective taken by Peleg and Yaari (1973) if we view the current consumer-investor as a leader in a leader follower game in which the followers are the same consumer-investor at future time periods. Current choices are best responses to the choices he expects to make in the future.

Our model uses the measure of risk aversion suggested by Kihlstrom and Mirman (1974, 1981). Preferences for consumption streams are CES and the elasticity of substitution can be chosen independently of the risk aversion measure. When the elasticity of substitution is the inverse of the risk aversion measure our model reduces to the additively separable model.

We refer to our approach as one of “consistent planning” by a “forward looking” expected utility maximizing consumer-investor. We introduce the approach and discuss the dynamic inconsistency in a three-period setting. We then describe the infinite horizon version of the model for the case in which risky asset returns are i.i.d. Since the standard additively separable model is a special case of ours we can easily compare that special case to our general model by simply noting the form our model takes when the elasticity of substitution equals the inverse of the risk aversion measure. Since our approach is an

alternative to the widely used non-expected utility, recursive approach of Epstein–Zin and Weil we also compare our model to theirs.

Finally we apply our approach to investigate the equilibrium equity premium obtained in the Lucas asset pricing model. We find that, in contrast to the Epstein–Zin and Weil approach, the equity premium obtained from our approach is affected by the elasticity of substitution as well as the risk aversion measure. Indeed, as Proposition 4 demonstrates, our model yields a higher equity premium than the additively separable model only when the elasticity of substitution in our model is exceeded by that of the additively separable model. Proposition 2 describes sufficient conditions under which expected utility is finite. Proposition 5 is an extension of a result obtained in Kihlstrom and Mirman (1974) and Diamond and Stiglitz (1974). We demonstrate that savings increase (decrease) with risk aversion if the elasticity of substitution is less (greater) than one.

We have yet to investigate the case of non-i.i.d. consumption growth. In recent work, Bansal and Yaron (2004) have shown that a model based on Epstein–Zin and Weil preferences can explain both the equity premium and the risk-free rate as well as other empirical regularities when consumption growth has “a long run predictable component” and there is some variation in the volatility of consumption growth. Thus, it seems clear that it is important to consider cases in which consumption growth is not i.i.d.

**Acknowledgements**

I would like to thank Andrew Abel, Andrew Postlewaite, Andreu Mas-Colell, Jacob Sagi, Raman Uppal, Amir Yaron and Stan Zin for their comments, suggestions and encouragement. I also want to thank my colleagues in the Finance and Economics Departments at the University of Pennsylvania for the comments they made during departmental seminars. In addition, I have benefited from the comments of participants in the Economics Department Seminars at the The University of Chicago, Cornell University, The University of Minnesota, Universitat Pompeu Fabra, Tel Aviv University and l’Université Toulouse. The paper was also presented at the Finance Department Seminar at The University of Melbourne and the Seminar at The Pacific Institute for the Mathematical Sciences of the University of British Columbia. I appreciate the helpful comments I received from the participants in those seminars as well as from the participants at the IMPA Workshop on Mathematical Economics in honor of Aloisio Araujo. Finally, I am particularly grateful to Narayana Kocherlakota whose suggestions and remarks strongly influenced my thinking on these issues and who provided the ideas for some of results in the paper.

**Appendix 1**

**Proof.** Proposition 1: Let

$$h(U, A^\rho) = \left[ \frac{1}{1-\alpha} \right] (A^\rho + U^\rho)^{(1-\alpha)/\rho}.$$

Then

$$R(U, A^\rho) = -\frac{h_{UU}(U, A^\rho)U}{h_U(U, A^\rho)} = \frac{[\alpha U^\rho + (1-\rho)A^\rho]}{(A^\rho + U^\rho)}.$$

When

$$1 - \rho > (<) \alpha$$

and

$$A > 0,$$

we have

$$R(U, A^\rho) = \frac{[\alpha U^\rho + (1-\rho)A^\rho]}{(A^\rho + U^\rho)} > (<) \alpha = R(U, 0).$$

Note that when commitment is impossible, the consumer-investor’s  $(c_2, \gamma_2)$  choice is

$$(c_2^*, \gamma_2^*)$$

which maximizes

$$Eh(\tilde{U}, 0) = E \left[ \frac{1}{1-\alpha} \right] \tilde{U}^{1-\alpha}$$

where

$$\tilde{U} = (c_2^\rho + \beta[1 - c_2]^\rho[r_f + \gamma_2 \tilde{x}_2]^\rho)^{1/\rho}.$$

When he can commit, his  $(c_2, \gamma_2)$  choice is

$$(\hat{c}_2(x_1), \hat{\gamma}_2(x_1))$$

which maximizes

$$Eh(\tilde{U}, A(x_1)) = \left[ \frac{1}{1-\alpha} \right] ([A(x_1)]^\rho + \tilde{U}^\rho)^{(1-\alpha)/\rho}$$

where again

$$\tilde{U} = (c_2^\rho + \beta[1 - c_2]^\rho[r_f + \gamma_2\bar{x}_2]^\rho)^{1/\rho}$$

and where

$$A(x_1) = \frac{\hat{c}_1}{\beta^{1/\rho}[1 - \hat{c}_1][r_f + \hat{\gamma}_1x_1]} > 0.$$

So, if

$$1 - \rho > (<) \alpha,$$

the consumer-investor acts as if he were more (less) risk averse in the Kihlstrom–Mirman sense when he chooses to commit to

$$(\hat{c}_2(x_1), \hat{\gamma}_2(x_1))$$

in the first period than he does when he chooses

$$(c_2^*, \gamma_2^*)$$

in the second period.

Also

$$\frac{\partial R(U, A^\rho)}{\partial A} = \frac{[1 - \rho - \alpha]\rho U^\rho A^{\rho-1}}{(A^\rho + U^\rho)^2} > (<) 0$$

when

$$1 - \rho > (<) \alpha$$

and

$$A > 0.$$

Since

$$A(x_1) = \frac{\hat{c}_1}{\beta^{1/\rho}[1 - \hat{c}_1][r_f + \hat{\gamma}_1x_1]}$$

$$A'(x_1) < 0,$$

and

$$\frac{\partial R(U, [A(x_1)]^\rho)}{\partial x_1} < (>) 0$$

when

$$1 - \rho > (<) \alpha.$$

This implies that, when

$$1 - \rho > (<) \alpha,$$

increases in  $x_1$  cause the consumer-investor to act as if he were less (more) risk averse in the Kihlstrom–Mirman sense when he commits to

$$(\hat{c}_2(x_1), \hat{\gamma}_2(x_1))$$

in the first period.  $\square$

**Appendix 2**

**Proof.** Proposition 2: For each  $T$ , let

$$f_T(\{g_{t+\tau}\}_{\tau=0}^\infty) = \left( c^\rho + \beta[1-c]^\rho \left( \frac{\hat{c}}{1-\hat{c}} \right)^\rho g_t^\rho \theta_T(\{g_{t+\tau}\}_{\tau=1}^\infty) \right)^{(1-\alpha)/\rho}$$

where

$$\theta_T(\{g_{t+\tau}\}_{\tau=1}^\infty) = \left[ 1 + \sum_{\tau=1}^T \beta^\tau \prod_{s=1}^\tau g_{t+s}^\rho \right].$$

Since

$$\begin{aligned} \theta_{T+1}(\{g_{t+\tau}\}_{\tau=1}^\infty) &= \left[ 1 + \sum_{\tau=1}^T \beta^\tau \prod_{s=1}^\tau g_{t+s}^\rho + \beta^{T+1} \prod_{s=1}^{T+1} g_{t+s}^\rho \right] = \theta_T(\{g_{t+\tau}\}_{\tau=1}^\infty) + \beta^{T+1} \prod_{s=1}^{T+1} g_{t+s}^\rho > \theta_T(\{g_{t+\tau}\}_{\tau=1}^\infty), \\ f_{T+1}(\{g_{t+\tau}\}_{\tau=0}^\infty) &= \left( c^\rho + \beta[1-c]^\rho \left( \frac{\hat{c}}{1-\hat{c}} \right)^\rho g_t^\rho \theta_{T+1}(\{g_{t+\tau}\}_{\tau=1}^\infty) \right)^{(1-\alpha)/\rho} \\ &> \left( c^\rho + \beta[1-c]^\rho \left( \frac{\hat{c}}{1-\hat{c}} \right)^\rho g_t^\rho \theta_T(\{g_{t+\tau}\}_{\tau=1}^\infty) \right)^{(1-\alpha)/\rho} = f_T(\{g_{t+\tau}\}_{\tau=0}^\infty). \end{aligned}$$

When condition (51) holds, Jensen’s inequality implies that

$$\begin{aligned} E[f_T(\{\tilde{g}_{t+\tau}\}_{\tau=0}^\infty)] &= E \left[ \left( c^\rho + \beta[1-c]^\rho \left( \frac{\hat{c}}{1-\hat{c}} \right)^\rho \tilde{g}_t^\rho \theta_T(\{\tilde{g}_{t+\tau}\}_{\tau=1}^\infty) \right)^{(1-\alpha)/\rho} \right] \\ &< \left( c^\rho + \beta[1-c]^\rho \left( \frac{\hat{c}}{1-\hat{c}} \right)^\rho E\tilde{g}_t^\rho E\theta_T(\{\tilde{g}_{t+\tau}\}_{\tau=1}^\infty) \right)^{(1-\alpha)/\rho}. \end{aligned} \tag{78}$$

Furthermore, for all  $T$ ,

$$\begin{aligned} \left( c^\rho + \beta[1-c]^\rho \left( \frac{\hat{c}}{1-\hat{c}} \right)^\rho E\tilde{g}_t^\rho E\theta_T(\{\tilde{g}_{t+\tau}\}_{\tau=1}^\infty) \right)^{(1-\alpha)/\rho} &< \left( c^\rho + \beta[1-c]^\rho \left( \frac{\hat{c}}{1-\hat{c}} \right)^\rho E\tilde{g}_t^\rho E\theta(\{\tilde{g}_{t+\tau}\}_{\tau=1}^\infty) \right)^{(1-\alpha)/\rho} \\ &= \left( c^\rho + \beta[1-c]^\rho \left( \frac{\hat{c}}{1-\hat{c}} \right)^\rho E\tilde{g}_t^\rho \left[ 1 + \sum_{\tau=1}^\infty (\beta E\tilde{g}^\rho)^\tau \right] \right)^{(1-\alpha)/\rho} < \infty \end{aligned} \tag{79}$$

if (52) holds. The monotone convergence theorem implies that

$$\begin{aligned} E \left[ \left( c^\rho + \beta[1-c]^\rho \left( \frac{\hat{c}}{1-\hat{c}} \right)^\rho \tilde{g}_t^\rho \theta(\{\tilde{g}_{t+\tau}\}_{\tau=1}^\infty) \right)^{(1-\alpha)/\rho} \right] &= \lim_{T \rightarrow \infty} E \left[ \left( c^\rho + \beta[1-c]^\rho \left( \frac{\hat{c}}{1-\hat{c}} \right)^\rho \tilde{g}_t^\rho \theta_T(\{\tilde{g}_{t+\tau}\}_{\tau=1}^\infty) \right)^{(1-\alpha)/\rho} \right] \\ &= \lim_{T \rightarrow \infty} E[f_T(\{\tilde{g}_{t+\tau}\}_{\tau=0}^\infty)]. \end{aligned}$$

Together (78) and (79) imply that, for all  $T$ ,

$$\begin{aligned} E \left[ \left( c^\rho + \beta[1-c]^\rho \left( \frac{\hat{c}}{1-\hat{c}} \right)^\rho \tilde{g}_t^\rho \theta_T(\{\tilde{g}_{t+\tau}\}_{\tau=1}^\infty) \right)^{(1-\alpha)/\rho} \right] \\ < \left( c^\rho + \beta[1-c]^\rho \left( \frac{\hat{c}}{1-\hat{c}} \right)^\rho E\tilde{g}_t^\rho \left[ 1 + \sum_{\tau=1}^\infty (\beta E\tilde{g}^\rho)^\tau \right] \right)^{(1-\alpha)/\rho} < \infty \end{aligned}$$



when (51) and (52) hold. Thus,

$$\begin{aligned} E \left[ \left( c^\rho + \beta[1-c]^\rho \left( \frac{\hat{c}}{1-\hat{c}} \right)^\rho \tilde{g}_t^\rho \theta(\{\tilde{g}_{t+\tau}\}_{\tau=1}^\infty) \right)^{(1-\alpha)/\rho} \right] \\ = \lim_{T \rightarrow \infty} E \left[ \left( c^\rho + \beta[1-c]^\rho \left( \frac{\hat{c}}{1-\hat{c}} \right)^\rho \tilde{g}_t^\rho \theta_T(\{\tilde{g}_{t+\tau}\}_{\tau=1}^\infty) \right)^{(1-\alpha)/\rho} \right] \\ < \left( c^\rho + \beta[1-c]^\rho \left( \frac{\hat{c}}{1-\hat{c}} \right)^\rho E \tilde{g}_t^\rho \left[ 1 + \sum_{\tau=1}^\infty (\beta E \tilde{g}^\rho)^\tau \right] \right)^{(1-\alpha)/\rho} < \infty \end{aligned}$$

if (51) and (52) hold. Since

$$E \left[ \left( c^\rho + \beta[1-c]^\rho \left( \frac{\hat{c}}{1-\hat{c}} \right)^\rho \tilde{g}_t^\rho \theta(\{\tilde{g}_{t+\tau}\}_{\tau=1}^\infty) \right)^{(1-\alpha)/\rho} \right]$$

is finite when (51) and (52) hold,

$$\theta(\{\tilde{g}_{t+\tau}\}_{\tau=1}^\infty) = \left[ 1 + \sum_{\tau=1}^\infty \beta^\tau \prod_{s=1}^\tau \tilde{g}_{t+s}^\rho \right]$$

must be finite almost surely under the same conditions.  $\square$

### Appendix 3

**Proof.** Proposition 4: This follows simply from the fact that

$$(1 + \beta g^\rho \theta)^{((1-\alpha)/\rho)-1} g^{\rho-1} \theta = (g^{-\rho} + \beta \theta)^{(1-\alpha-\rho)/\rho} g^{-\alpha} \theta$$

which implies that

$$\frac{E[(1 + \beta \tilde{g}^\rho \tilde{\theta})^{((1-\alpha)/\rho)-1} \tilde{g}^\rho \tilde{\theta}]}{E[(1 + \beta \tilde{g}^\rho \tilde{\theta})^{((1-\alpha)/\rho)-1} \tilde{g}^{\rho-1} \tilde{\theta}]} = \frac{E[(\tilde{g}^{-\rho} + \beta \tilde{\theta})^{(1-\alpha-\rho)/\rho} \tilde{g}^{1-\alpha} \tilde{\theta}]}{E[(\tilde{g}^{-\rho} + \beta \tilde{\theta})^{(1-\alpha-\rho)/\rho} \tilde{g}^{-\alpha} \tilde{\theta}]} \quad (80)$$

When (70) holds

$$g < (>) \frac{E \tilde{g}^{1-\alpha}}{E \tilde{g}^{-\alpha}}$$

implies

$$(g^{-\rho} + \beta \theta)^{(1-\alpha-\rho)/\rho} > (<) \left( \left[ \frac{E \tilde{g}^{1-\alpha}}{E \tilde{g}^{-\alpha}} \right]^{-\rho} + \beta \theta \right)^{(1-\alpha-\rho)/\rho}$$

So, for each  $\theta$  and  $g$ ,

$$(g^{-\rho} + \beta \theta)^{(1-\alpha-\rho)/\rho} \left( g - \frac{E \tilde{g}^{1-\alpha}}{E \tilde{g}^{-\alpha}} \right) \theta < \left( \left[ \frac{E \tilde{g}^{1-\alpha}}{E \tilde{g}^{-\alpha}} \right]^{-\rho} + \beta \theta \right)^{(1-\alpha-\rho)/\rho} \left( g - \frac{E \tilde{g}^{1-\alpha}}{E \tilde{g}^{-\alpha}} \right) \theta.$$

Multiplying each side of this equation by

$$\frac{g^{-\alpha}}{E \tilde{g}^{-\alpha}}$$

and taking expectations over  $\tilde{g}$ , we then have

$$E \left[ (\tilde{g}^{-\rho} + \beta \theta)^{((1-\alpha-\rho)/\rho)} \left( \frac{\tilde{g}^{1-\alpha}}{E \tilde{g}^{-\alpha}} - \frac{\tilde{g}^{-\alpha}}{E \tilde{g}^{-\alpha}} \left[ \frac{E \tilde{g}^{1-\alpha}}{E \tilde{g}^{-\alpha}} \right] \right) \theta \right] < E \left[ \left( \left[ \frac{E \tilde{g}^{1-\alpha}}{E \tilde{g}^{-\alpha}} \right]^{-\rho} + \beta \theta \right)^{((1-\alpha-\rho)/\rho)} \left( \frac{E \tilde{g}^{1-\alpha}}{E \tilde{g}^{-\alpha}} - \frac{E \tilde{g}^{1-\alpha}}{E \tilde{g}^{-\alpha}} \right) \theta \right] = 0.$$

Thus, for each  $\theta$ ,

$$\frac{E[(\tilde{g}^{-\rho} + \beta \theta)^{(1-\alpha-\rho)/\rho} \tilde{g}^{1-\alpha} \theta]}{E[(\tilde{g}^{-\rho} + \beta \theta)^{(1-\alpha-\rho)/\rho} \tilde{g}^{-\alpha} \theta]} < \frac{E \tilde{g}^{1-\alpha}}{E \tilde{g}^{-\alpha}} \quad (81)$$

Taking expectations over  $\theta$  and using (80), (81) implies (69). Note that when (70) is reversed a parallel argument implies that the inequality (69) is reversed.  $\square$

**Appendix 4**

**Proof.** Proposition 5: Assume that

$$\alpha_2 < \alpha_1.$$

Use the fact that, for each  $\theta$ ,

$$(1 + \beta g^\rho \theta)^{((1-\alpha_1)/\rho)-1} = (1 + \beta g^\rho \theta)^{((\alpha_2-\alpha_1)/\rho)} (1 + \beta g^\rho \theta)^{((1-\alpha_2)/\rho)-1}.$$

If  $\rho > (<) 0$ , then

$$(1 + \beta g^\rho \theta)^{(\alpha_2-\alpha_1)/\rho} < (>) \left( 1 + \left[ \frac{1 - \hat{c}(\alpha_2)}{\hat{c}(\alpha_2)} \right] \right)^{(\alpha_2-\alpha_1)/\rho}$$

when

$$1 < \beta \left[ \frac{\hat{c}(\alpha_2)}{1 - \hat{c}(\alpha_2)} \right] g^\rho \theta$$

and

$$(1 + \beta g^\rho \theta)^{(\alpha_2-\alpha_1)/\rho} > (<) \left( 1 + \left[ \frac{1 - \hat{c}(\alpha_2)}{\hat{c}(\alpha_2)} \right] \right)^{(\alpha_2-\alpha_1)/\rho}$$

when

$$1 > \beta \left[ \frac{\hat{c}(\alpha_2)}{1 - \hat{c}(\alpha_2)} \right] g^\rho \theta.$$

So, for all  $g$  and  $\theta$

$$\begin{aligned} & (1 + \beta g^\rho \theta)^{(\alpha_2-\alpha_1)/\rho} (1 + \beta g^\rho \theta)^{((1-\alpha_2)/\rho)-1} \left[ 1 - \beta \left[ \frac{\hat{c}(\alpha_2)}{1 - \hat{c}(\alpha_2)} \right] g^\rho \theta \right] \\ & > (<) \left( 1 + \left[ \frac{1 - \hat{c}(\alpha_2)}{\hat{c}(\alpha_2)} \right] \right)^{(\alpha_2-\alpha_1)/\rho} (1 + \beta g^\rho \theta)^{((1-\alpha_2)/\rho)-1} \left[ 1 - \beta \left[ \frac{\hat{c}(\alpha_2)}{1 - \hat{c}(\alpha_2)} \right] g^\rho \theta \right] \end{aligned}$$

if  $\rho > (<) 0$ . Thus, when  $\rho > (<) 0$ , we have

$$\begin{aligned} & E(1 + \beta \tilde{g}^\rho \tilde{\theta})^{((1-\alpha_1)/\rho)-1} \left[ 1 - \beta \left[ \frac{\hat{c}(\alpha_2)}{1 - \hat{c}(\alpha_2)} \right] \tilde{g}^\rho \tilde{\theta} \right] \\ & = E(1 + \beta \tilde{g}^\rho \tilde{\theta})^{((\alpha_2-\alpha_1)/\rho)} (1 + \beta \tilde{g}^\rho \tilde{\theta})^{((1-\alpha_2)/\rho)-1} \left[ 1 - \beta \left[ \frac{\hat{c}(\alpha_2)}{1 - \hat{c}(\alpha_2)} \right] \tilde{g}^\rho \tilde{\theta} \right] \\ & > (<) \left( 1 + \left[ \frac{1 - \hat{c}(\alpha_2)}{\hat{c}(\alpha_2)} \right] \right)^{((\alpha_2-\alpha_1)/\rho)} E(1 + \beta \tilde{g}^\rho \tilde{\theta})^{((1-\alpha_2)/\rho)-1} \left[ 1 - \beta \left[ \frac{\hat{c}(\alpha_2)}{1 - \hat{c}(\alpha_2)} \right] \tilde{g}^\rho \tilde{\theta} \right] \\ & = 0 = E(1 + \beta \tilde{g}^\rho \tilde{\theta})^{((1-\alpha_1)/\rho)-1} \left[ 1 - \beta \left[ \frac{\hat{c}(\alpha_1)}{1 - \hat{c}(\alpha_1)} \right] \tilde{g}^\rho \tilde{\theta} \right]. \end{aligned}$$

This implies that

$$\left[ \frac{\hat{c}(\alpha_1)}{1 - \hat{c}(\alpha_1)} \right] > (<) \left[ \frac{\hat{c}(\alpha_2)}{1 - \hat{c}(\alpha_2)} \right]$$

if  $\rho > (<) 0$ .  $\square$

**Appendix 5**

**Proof.** Proposition 7: Note that

$$(1 + \beta g^\rho \theta)^{((1-\alpha_1)/\rho)-1} = (1 + \beta g^\rho \theta)^{(\alpha_2-\alpha_1)/\rho} (1 + \beta g^\rho \theta)^{((1-\alpha_2)/\rho)-1}.$$

When

$$(1 + \beta g^\rho \theta)^{(\alpha_2 - \alpha_1)/\rho}$$

is a decreasing function of  $g$ . Thus,

$$g > (<) E_2^* \tilde{g}$$

implies

$$(1 + \beta g^\rho \theta)^{(\alpha_2 - \alpha_1)/\rho} < (>) (1 + \beta (E_2^* \tilde{g})^\rho \theta)^{(\alpha_2 - \alpha_1)/\rho}.$$

We then have, for all  $g$  and  $\theta$ ,

$$(1 + \beta g^\rho \theta)^{((1 - \alpha_1)/\rho) - 1} \theta g^{\rho - 1} (g - E_2^* \tilde{g}) = (1 + \beta g^\rho \theta)^{(\alpha_2 - \alpha_1)/\rho} (1 + \beta g^\rho \theta)^{((1 - \alpha_2)/\rho) - 1} \theta g^{\rho - 1} (g - E_2^* \tilde{g}) < (1 + \beta (E_2^* \tilde{g})^\rho \theta)^{(\alpha_2 - \alpha_1)/\rho} (1 + \beta g^\rho \theta)^{((1 - \alpha_2)/\rho) - 1} \theta g^{\rho - 1} (g - E_2^* \tilde{g}).$$

Taking expectations over  $g$ ,

$$E[(1 + \beta \tilde{g}^\rho \theta)^{((1 - \alpha_1)/\rho) - 1} \theta \tilde{g}^{\rho - 1} (\tilde{g} - E_2^* \tilde{g})] < (1 + \beta (E_2^* \tilde{g})^\rho \theta)^{(\alpha_2 - \alpha_1)/\rho} E[(1 + \beta \tilde{g}^\rho \theta)^{((1 - \alpha_2)/\rho) - 1} \theta \tilde{g}^{\rho - 1} (\tilde{g} - E_2^* \tilde{g})] \tag{82}$$

for each  $\theta$ . Then taking expectations over  $\tilde{\theta}$  we have

$$E[(1 + \beta \tilde{g}^\rho \tilde{\theta})^{((1 - \alpha_1)/\rho) - 1} \tilde{\theta} \tilde{g}^{\rho - 1} (\tilde{g} - E_2^* \tilde{g})] < E[(1 + \beta (E_2^* \tilde{g})^\rho \tilde{\theta})^{(\alpha_2 - \alpha_1)/\rho} [(1 + \beta \tilde{g}^\rho \tilde{\theta})^{((1 - \alpha_2)/\rho) - 1} \tilde{\theta} \tilde{g}^{\rho - 1} (\tilde{g} - E_2^* \tilde{g})]]. \tag{83}$$

We will have (71) if we can demonstrate that

$$E[(1 + \beta (E_2^* \tilde{g})^\rho \tilde{\theta})^{(\alpha_2 - \alpha_1)/\rho} [(1 + \beta \tilde{g}^\rho \tilde{\theta})^{((1 - \alpha_2)/\rho) - 1} \tilde{\theta} \tilde{g}^{\rho - 1} (\tilde{g} - E_2^* \tilde{g})]] < 0. \tag{84}$$

Now let  $\theta_0$  be defined by

$$E[(1 + \beta \tilde{g}^\rho \theta_0)^{((1 - \alpha_2)/\rho) - 1} \theta_0 \tilde{g}^{\rho - 1} (\tilde{g} - E_2^* \tilde{g})] = 0.$$

Then

$$\frac{E[(1 + \beta \tilde{g}^\rho \theta_0)^{((1 - \alpha_2)/\rho) - 1} \tilde{g}^\rho]}{E[(1 + \beta \tilde{g}^\rho \theta_0)^{((1 - \alpha_2)/\rho) - 1} \tilde{g}^{\rho - 1}]} = E_2^* \tilde{g}.$$

We will show below that

$$1 - \alpha_2 > (<) \rho$$

implies that

$$\frac{E[(1 + \beta \tilde{g}^\rho \theta)^{((1 - \alpha_2)/\rho) - 1} \tilde{g}^\rho]}{E[(1 + \beta \tilde{g}^\rho \theta)^{((1 - \alpha_2)/\rho) - 1} \tilde{g}^{\rho - 1}]}$$

is an increasing (decreasing) function of  $\theta$ . Thus, when

$$1 - \alpha_2 > \rho$$

$$\frac{E[(1 + \beta \tilde{g}^\rho \theta)^{((1 - \alpha_2)/\rho) - 1} \tilde{g}^\rho]}{E[(1 + \beta \tilde{g}^\rho \theta)^{((1 - \alpha_2)/\rho) - 1} \tilde{g}^{\rho - 1}]} > (<) E_2^* \tilde{g} = \frac{E[(1 + \beta \tilde{g}^\rho \theta_0)^{((1 - \alpha_2)/\rho) - 1} \tilde{g}^\rho]}{E[(1 + \beta \tilde{g}^\rho \theta_0)^{((1 - \alpha_2)/\rho) - 1} \tilde{g}^{\rho - 1}]}$$

and

$$E[(1 + \beta \tilde{g}^\rho \theta)^{((1 - \alpha_2)/\rho) - 1} \theta \tilde{g}^{\rho - 1} (\tilde{g} - E_2^* \tilde{g})] > (<) 0$$

if

$$\theta > (<) \theta_0.$$

When

$$1 - \alpha_2 < \rho$$

we have

$$\frac{E[(1 + \beta \tilde{g}^\rho \theta)^{((1 - \alpha_2)/\rho) - 1} \tilde{g}^\rho]}{E[(1 + \beta \tilde{g}^\rho \theta)^{((1 - \alpha_2)/\rho) - 1} \tilde{g}^{\rho - 1}]} < (>) E_2^* \tilde{g} = \frac{E[(1 + \beta \tilde{g}^\rho \theta_0)^{((1 - \alpha_2)/\rho) - 1} \tilde{g}^\rho]}{E[(1 + \beta \tilde{g}^\rho \theta_0)^{((1 - \alpha_2)/\rho) - 1} \tilde{g}^{\rho - 1}]}$$

and

$$E[(1 + \beta \tilde{g}^\rho \theta)^{((1-\alpha_2)/\rho)-1} \theta \tilde{g}^{\rho-1} (\tilde{g} - E_2^* \tilde{g})] < (>) 0$$

if

$$\theta > (<) \theta_0.$$

If we also assume that

$$\rho < (>) 0$$

then the fact that

$$\alpha_2 < \alpha_1$$

implies that

$$(1 + \beta(E_2^* \tilde{g})^\rho \theta)^{(\alpha_2 - \alpha_1)/\rho}$$

is an increasing (decreasing) function of  $\theta$ . Thus, when

$$\rho < 0$$

and

$$1 - \alpha_2 < \rho$$

or when

$$\rho > 0$$

and

$$1 - \alpha_2 > \rho$$

the inequality

$$(1 + \beta(E_2^* \tilde{g})^\rho \theta)^{(\alpha_2 - \alpha_1)/\rho} E[(1 + \beta \tilde{g}^\rho \theta)^{((1-\alpha_2)/\rho)-1} \theta \tilde{g}^{\rho-1} (\tilde{g} - E_2^* \tilde{g})] < (1 + \beta(E_2^* \tilde{g})^\rho \theta_0)^{(\alpha_2 - \alpha_1)/\rho} E[(1 + \beta \tilde{g}^\rho \theta)^{((1-\alpha_2)/\rho)-1} \theta \tilde{g}^{\rho-1} (\tilde{g} - E_2^* \tilde{g})]$$

holds for all  $\theta$ . Taking expectations over  $\theta$

$$E[(1 + \beta(E_2^* \tilde{g})^\rho \tilde{\theta})^{(\alpha_2 - \alpha_1)/\rho} [(1 + \beta \tilde{g}^\rho \tilde{\theta})^{((1-\alpha_2)/\rho)-1} \tilde{\theta} \tilde{g}^{\rho-1} (\tilde{g} - E_2^* \tilde{g})]] < (1 + \beta(E_2^* \tilde{g})^\rho \theta_0)^{(\alpha_2 - \alpha_1)/\rho} E[(1 + \beta \tilde{g}^\rho \tilde{\theta})^{((1-\alpha_2)/\rho)-1} \tilde{\theta} \tilde{g}^{\rho-1} (\tilde{g} - E_2^* \tilde{g})]. \tag{85}$$

The right-hand side of the inequality in (85) is 0 by definition of  $E_2^* \tilde{g}$ . So (84) holds if

$$\rho < 0$$

and

$$1 - \alpha_2 < \rho$$

or

$$\rho > 0$$

and

$$1 - \alpha_2 > \rho.$$

The proof is complete if we can show that

$$\frac{E[(1 + \beta \tilde{g}^\rho \theta)^{((1-\alpha_2)/\rho)-1} \tilde{g}^\rho]}{E[(1 + \beta \tilde{g}^\rho \theta)^{((1-\alpha_2)/\rho)-1} \tilde{g}^{\rho-1}]}$$

is increasing (decreasing) in  $\theta$  when

$$1 - \alpha_2 > (<) \rho.$$

Using a generalization of the argument employed above to obtain (82) we can show that

$$\frac{Eu_g(g; \theta)\tilde{g}}{Eu_g(g; \theta)}$$

is increasing in  $\theta$  if the absolute risk aversion measure

$$-\frac{u_{gg}(g; \theta)}{u_g(g; \theta)}$$

is decreasing in  $\theta$ . Note that when

$$u(g; \theta) = (1 + \beta g^\rho \theta)^{(1-\alpha_2)/\rho},$$

the risk aversion measure

$$-\frac{u_{gg}(g; \theta)}{u_g(g; \theta)} = \frac{g(1 - \alpha_2 - \rho)}{(1 + \beta g^\rho \theta)} + \alpha_2 g$$

is decreasing (increasing) in  $\theta$  when

$$1 - \alpha_2 > (<) \rho.$$

Thus,

$$\frac{Eu_g(g; \theta)\tilde{g}}{Eu_g(g; \theta)} = \frac{E[(1 + \beta \tilde{g}^\rho \theta)^{(1-\alpha_2)/\rho - 1} \tilde{g}^\rho]}{E[(1 + \beta \tilde{g}^\rho \theta)^{(1-\alpha_2)/\rho - 1} \tilde{g}^{\rho-1}]}$$

is increasing (decreasing) in  $\theta$  when

$$1 - \alpha_2 > (<) \rho.$$

□

## References

- Abel, A., 1990. Asset prices under habit formation and catching up with the Joneses. *American Economic Review* 80, 38–42.
- Abel, A., 1999. Risk premia and term premia in general equilibrium. *Journal of Monetary Economics* 43, 3–33.
- Arrow, K.J., 1971. *Essays in the Theory of Risk Bearing*. Markham, Chicago.
- Bansal, R., Yaron, A., 2004. Risks for the long run: a potential resolution of asset pricing puzzles. *Journal of Finance* 59, 1481–1509.
- Bhamra, H., Uppal, R., 2006. The role of risk aversion and intertemporal substitution in dynamic consumption–portfolio choice with recursive utility. *Journal of Economic Dynamics and Control* 30, 967–991.
- Breeden, D., 1979. An intertemporal asset pricing model with stochastic consumption and investment opportunities. *Journal of Financial Economics* 7, 265–296.
- Bulow, J., 1982. Durable goods monopolists. *Journal of Political Economy* 90, 314–332.
- Campbell, J., Cochrane, J., 1999. By force of habit: a consumption-based explanation of aggregate stock market behavior. *Journal of Political Economy* 107, 205–251.
- Coase, R., 1972. Durability and monopoly. *Journal of Law and Economics* 15, 143–149.
- Constantinides, G., 1990. Habit formation: a resolution of the equity premium puzzle. *Journal of Political Economy* 98, 519–543.
- Debreu, G., 1976. Least concave utility functions. *Journal of Mathematical Economics* 3, 121–129.
- Diamond, P., Stiglitz, J., 1974. Increasing risk and risk aversion. *Journal of Economic Theory* 8, 337–360.
- Eden, B., August 2005. Substitution and risk aversion: is risk aversion important for understanding asset prices? Working Paper. Department of Economics, Vanderbilt University.
- Epstein, L., 1988. Risk aversion and asset prices. *Journal of Monetary Economics* 22, 179–192.
- Epstein, L., Zin, S., 1989. Substitution, risk aversion and the temporal behavior of consumption and asset returns: a theoretical framework. *Econometrica* 57, 937–969.
- Epstein, L., Zin, S., 1990. 'First-order' risk aversion and the equity premium puzzle. *Journal of Monetary Economics* 26, 387–407.
- Epstein, L., Zin, S., 1991. Substitution, risk aversion, and the temporal behavior of consumption and asset returns: an empirical analysis. *Journal of Political Economy* 99, 263–286.
- Grossman, S., Shiller, R., 1981. The determinants of the variability of stock market prices. *American Economic Review, Papers and Proceedings* 71, 222–227.
- Grossman, S., Melino, A., Shiller, R., 1987. Estimating the continuous-time consumption-based asset-pricing model. *Journal of Business and Economic Statistics* 5, 315–327.
- Hakansson, N., 1970. Optimal investment and consumption strategies under risk for a class of utility functions. *Econometrica* 38, 587–607.
- Hansen, L.P., Singleton, K., 1983. Stochastic consumption, risk aversion and the temporal behavior of asset returns. *Journal of Political Economy* 91, 249–265.
- Harris, C., Laibson, D., 2001. Dynamic choices of hyperbolic consumers. *Econometrica* 69, 935–957.
- Kihlstrom, R., Mirman, L., 1974. Risk aversion with many commodities. *Journal of Economic Theory* 8, 361–388.
- Kihlstrom, R., Mirman, L., 1981. Constant, increasing and decreasing risk aversion with many commodities. *Review of Economic Studies* 48, 271–280.
- Kocherlakota, N., 1990. Disentangling the coefficient of relative risk aversion from the intertemporal elasticity of substitution. *Journal of Finance* 45, 175–190.
- Kreps, D.M., Porteus, E.L., 1978. Temporal resolution of uncertainty and dynamic choice theory. *Econometrica* 46, 185–200.
- Kreps, D.M., Porteus, E.L., 1979a. Temporal von Neumann Morgenstern and induced preferences. *Journal of Economic Theory* 20, 81–109.
- Kreps, D.M., Porteus, E.L., 1979b. Dynamic choice theory and dynamic programming. *Econometrica* 47, 91–100.
- Labadie, 1986. Comparative dynamics and risk premia in an overlapping generations model. *The Review of Economic Studies* 53, 139–152.
- Laibson, D., 1997. Golden eggs and hyperbolic discounting. *Quarterly Journal of Economics* 112, 443–477.
- Levhari, D., Srinivasan, T.N., 1969. Optimal savings under uncertainty. *The Review of Economic Studies* 36, 153–163.
- Lucas, R., 1978. Asset prices in an exchange economy. *Econometrica* 46, 1429–1445.
- Luttmer, E., Mariotti, T., 2003. Subjective discounting in an exchange economy. *Journal of Political Economy* 111, 959–989.

- Luttmer, E., Mariotti, T., 2007. Efficiency and equilibrium when preferences are time inconsistent. *Journal of Economic Theory* 132, 493–506.
- Merton, R., 1969. Lifetime portfolio selection under uncertainty: the continuous time case. *The Review of Economics and Statistics* 51, 247–257.
- Merton, R., 1971. Optimal consumption and portfolio rules in a continuous-time model. *Journal of Economic Theory* 3, 373–413.
- Merton, R., 1973. An intertemporal capital asset pricing model. *Econometrica* 41, 867–887.
- Mehra, R., Prescott, E., 1985. The equity premium: a puzzle. *Journal of Monetary Economics* 15, 145–161.
- Peleg, B., Yaari, M., 1973. On the existence of a consistent course of action when tastes are changing. *Review of Economic Studies* 40, 391–401.
- Phelps, E.S., Pollak, R., 1968. On second best national saving and game-equilibrium growth. *Review of Economic Studies* 35, 185–199.
- Pollak, R., 1968. Consistent planning. *Review of Economic Studies* 35, 201–208.
- Pratt, J., 1964. Risk aversion in the large and in the small. *Econometrica* 32, 122–136.
- Samuelson, P., 1969. Lifetime portfolio selection by dynamic stochastic programming. *The Review of Economics and Statistics* 51, 239–246.
- Stokey, N., 1979. Intertemporal price discrimination. *Quarterly Journal of Economics* 93, 355–371.
- Strotz, R., 1956. Myopia and inconsistency in dynamic utility maximization. *Review of Economic Studies* 23, 165–180.
- Swenson, L.E.O., 1989. Portfolio choice with non-expected utility in continuous time. *Economic Letters* 30, 313–317.
- Van den Heuvel, S., October 2007. Temporal risk aversion and asset prices. Working Paper. Department of Finance, The Wharton School, University of Pennsylvania.
- Weil, P., 1989. The equity premium puzzle and the risk free rate puzzle. *Journal of Monetary Economics* 24, 401–421.
- Weil, P., 1990. Nonexpected utility in macroeconomics. *Quarterly Journal of Economics* 105, 29–42.